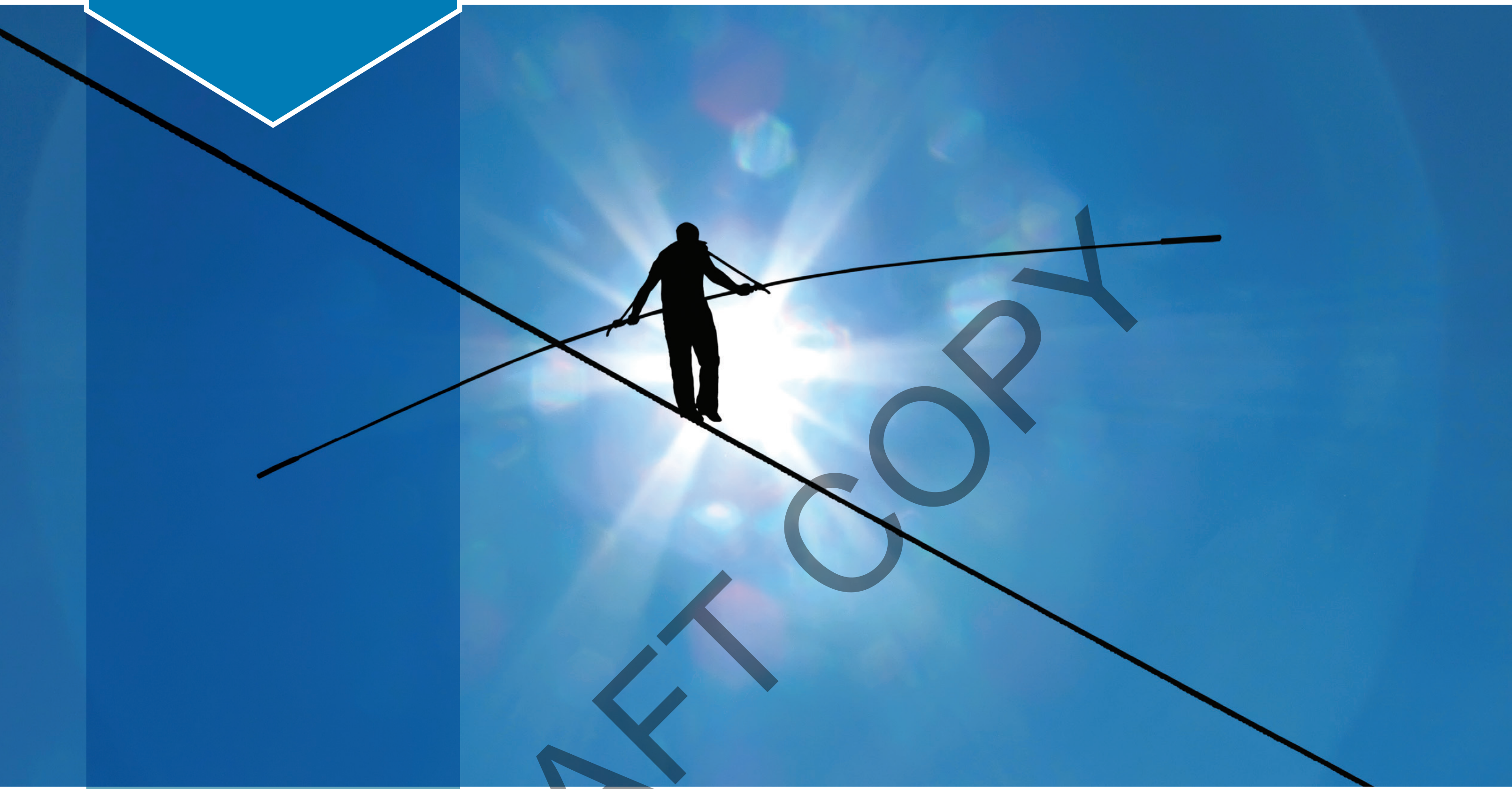
Contents

3 Forces and motion



In the beginning (if there was such a thing) God created Newton's laws of motion together with the necessary masses and forces. This is all; everything beyond this follows from the development of appropriate mathematical methods by means of deduction.

Albert Einstein

→ This tightrope walker is stationary. There are a number of forces acting on the walker which cancel each other out, resulting in no motion. In order for that to be possible, the cable must make small angles to the horizontal so that the vertical components of the tension can cancel out the weight of the walker. In that case, the tensions in the cable will be greater than the walker's weight.

1 Forces and Newton's laws of motion

Modelling vocabulary

Mechanics is about modelling the real world. In order to do this, suitable simplifying assumptions are often made so that mathematics can be applied to situations and problems. This process involves identifying factors that can be neglected without losing too much accuracy. Here are some commonly used modelling terms which are used to describe such assumptions:

- negligible: small enough to ignore
- inextensible: for a string with negligible stretch
- light: for an object with negligible mass
- particle: an object with negligible dimensions
- smooth: for a surface with negligible friction
- uniform: the same throughout.

Forces

A force is defined as the physical quantity that causes a change in motion. As it depends on magnitude and direction, it is a vector quantity.

Forces can start motion, stop motion, speed up or slow down objects, or change the direction of their motion. In real situations, several forces usually act on an object. The sum of these forces, known as the resultant force, determines whether or not there is a change of motion.

There are several types of force that you often use.

Note

g varies around the world, with $9.80\,\mathrm{m\,s^{-2}}$ being a typical value. Singapore, at 9.766, has one of the lowest values, and Helsinki, with 9.825, has one of the highest. You need to look carefully at the level of accuracy to which g is given in any problem. Your final answer should always reflect the accuracy of the given information.

The force of gravity

Every object on or near the Earth's surface is pulled vertically downwards by the force of gravity. The size of the force on an object of mass M kg is Mg newtons where g is a constant whose value is about $9.8\,\mathrm{m\,s^{-2}}$. The force of gravity is also known as the **weight** of the object.

Tension and thrust

When a string is pulled, as in Figure 3.1, it exerts a **tension** force opposite to the pull. The tension acts along the string and is the same throughout the string. A rigid rod can exert a tension force in a similar way to a string when it is used to support or pull an object. It can also exert a **thrust** force when it is in compression, as in Figure 3.2. The thrust acts along the rod and is the same throughout the rod.

The tension on either side of a smooth pulley is the same, as shown in Figure 3.3.

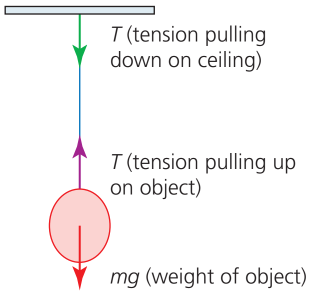


Figure 3.1

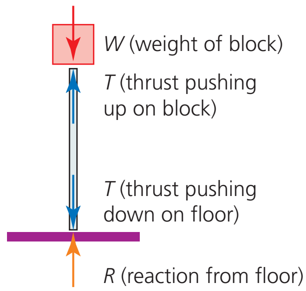


Figure 3.2

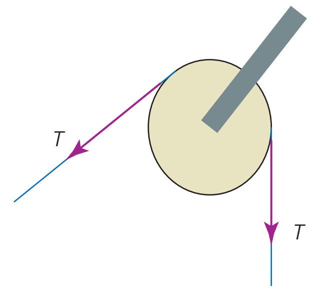
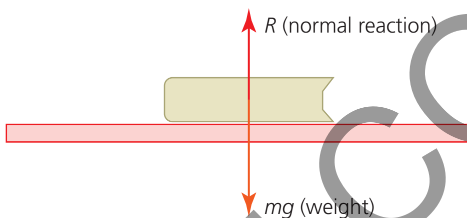


Figure 3.3

Normal reaction

A book resting on a table is subjected to two forces, its weight and the **normal reaction** of the table. It is called normal because its line of action is normal (at right angles) to the surface of the table. Since the book is in equilibrium, the normal reaction is equal and opposite to the weight of the book; it is a positive force.

> **Note**
>
> In Figure 3.4, the normal reaction is vertical but this is not always the case. For example, the normal reaction on an object on a slope is perpendicular to the slope.



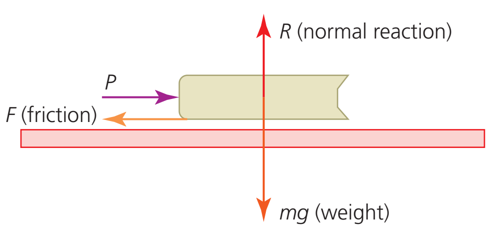
The normal reaction force is also often called the normal contact force.

Figure 3.4

If the book is about to lose contact with the table (which might happen, for instance, if the table is accelerating rapidly downwards), the normal force becomes zero.

Frictional force

In this diagram, the book on the table is being pushed by a force P parallel to the surface. The book remains at rest because P is balanced by a **frictional force**, F, in the opposite direction to P. The magnitude of the frictional force is equal to the pushing force $P = F$



The frictional force is also sometimes called frictional contact force.

Figure 3.5

If P is increased and the book starts to move, F is still present but now $P > F$. Friction always acts in the opposite direction to the motion. Friction may prevent the motion of an object or slow it down if it is moving.

Driving force

In problems about moving objects such as cars, all the forces acting along the line of motion can usually be reduced to two or three: the **driving force**, the **resistance** to motion and, possibly, a **braking force**.

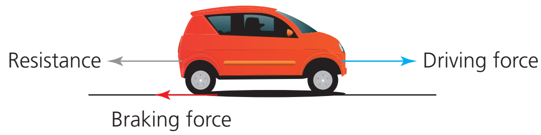


Figure 3.6

Example 3.1

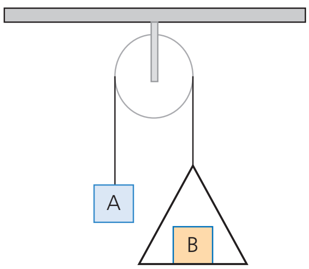


Figure 3.7

Figure 3.7 shows a block A of mass 10 kg connected to a light scale pan by a light inextensible string that passes over a light smooth pulley. The scale pan holds block B, also of mass 10 kg. The system is in equilibrium.

(i) On separate diagrams, show all the forces acting on each of the masses, the scale pan and the pulley.

(ii) Find the value of the tension in the string.

(iii) Find the tension in the rod holding the pulley.

(iv) Find the normal reaction of B on the scale pan.

Solution

(i)

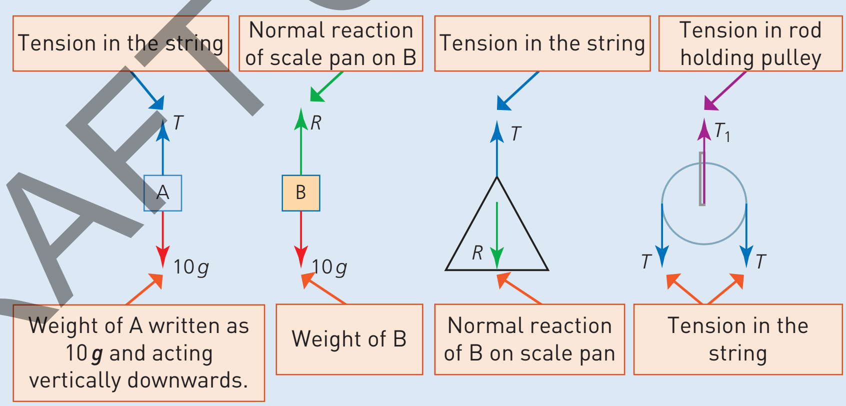


Figure 3.8

(ii) Block A is in equilibrium: $\Rightarrow$ $T = 10g = 98$ N.

(iii) The pulley is in equilibrium: $\Rightarrow$ $T_1 = 2T = 196$ N.

(iv) Block B is in equilibrium: $\Rightarrow$ $R = 10g = 98$ N.

Newton's laws of motion

1 Every object continues in a state of rest or uniform motion in a straight line unless it is acted on by a resultant external force.

2 The acceleration of an object is proportional to, and in the same direction as, the resultant of the forces acting on the object.

$$\mathbf{F} = m\mathbf{a}$$

F is the resultant force.
m is the mass of the object.
a is the acceleration.

Historical note

Isaac Newton was born in Lincolnshire in 1642. He was not an outstanding scholar either as a schoolboy or as a university student, yet later in life he made remarkable contributions in dynamics, optics, astronomy, chemistry, music theory and theology. He became Member of Parliament for Cambridge University and later Warden of the Royal Mint. His tomb in Westminster Abbey reads 'Let mortals rejoice that there existed such and so great an ornament to the Human Race'.

Notice that this is a vector equation, since both the magnitudes and directions of the resultant force and the acceleration are involved. If the motion is along a straight line it is often written in scalar form as $F = ma$.

3 When one object exerts a force on another there is always a reaction, which is equal and opposite in direction to the acting force.

Equation of motion

The equation resulting from Newton's second law is often described as an **equation of motion**, as in the following examples.

Example 3.2

An empty bottle of mass 0.5 kg is released from a submarine and rises with an acceleration of $0.75\,\text{m s}^{-2}$. The water causes a resistance of 1.1 N.

(i) Draw a diagram showing the forces acting on the bottle and the direction of its acceleration.

(ii) Write down the equation of motion of the bottle.

(iii) Find the size of the buoyancy force.

Solution

(i) The forces acting on the bottle and the acceleration are shown in this diagram.

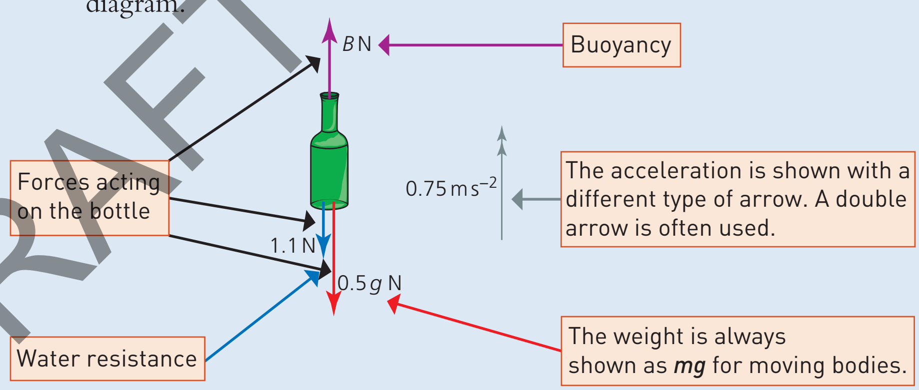


Figure 3.9

(ii) The resultant force acting on the bottle is $(B - 0.5g - 1.1)$ upwards.

The resulting equation

$$B - 0.5g - 1.1 = 0.5a$$

is called the **equation of motion**.

(iii) $B - 4.9 - 1.1 = 0.375$ ← 0.5×0.75

$$B = 6.375$$

The buoyancy force on the bottle is 6.4 N.

Example 3.3

A car of mass 900 kg travels at a constant speed of $20\,\mathrm{m\,s^{-1}}$ along a straight horizontal road. Its engine is producing a driving force of 500 N.

(i) What is the resistance to its motion?

Later the driving force is removed and the car is brought to rest in a time of 5 s with the same resistance to motion.

(ii) Find the force created by the brakes, assuming it to be constant.

Solution

(i) The car is travelling at constant speed, so that the resultant force acting on the car is zero.

Resultant = 500 – R

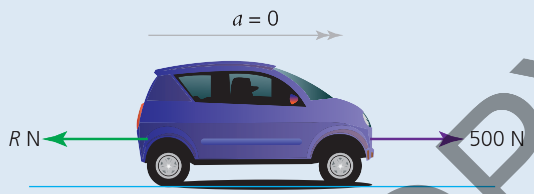


Figure 3.10

Let the resistive force be R N.

$$500 - R = 0$$
$$R = 500$$

The resistive force is 500 N.

(ii) The car is slowing down.

So you expect a to be negative.

Resultant = 500 – B

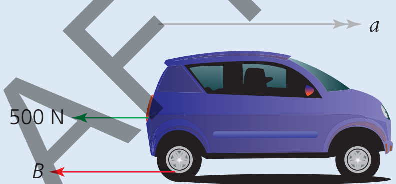


Figure 3.11

B is constant, so that a is also constant and you can use the constant acceleration formulae.

The equation of motion is $-B - 500 = 900a$ ①

$0 = 20 + a \times 5$

Use $v = u + at$ with $u = 20$, $v = 0$ and $t = 5$.

$a = -4\,\mathrm{m\,s^{-2}}$

Substituting in ① $-B - 500 = 900 \times -4$

$\Rightarrow B = 3100\,\mathrm{N}$

The braking force is 3100 N.

Example 3.4

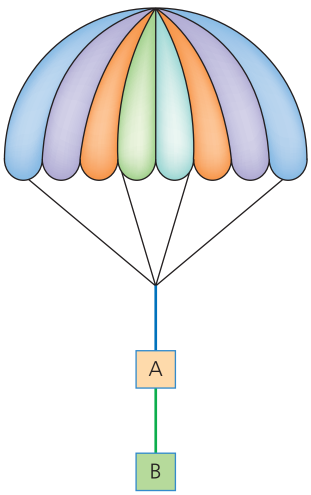


Figure 3.12

Two boxes A and B are descending vertically supported by a parachute. Box A has mass 100 kg. Box B has mass 75 kg and is suspended from box A by a light vertical wire. Both boxes are descending with acceleration $3\,\text{m}\,\text{s}^{-2}$.

(i) Draw a labelled diagram showing all the forces acting on box A and another diagram showing all the forces acting on box B.

(ii) Write down separate equations of motion for box A and for box B.

(iii) Find the tensions in both wires.

Solution

(i)

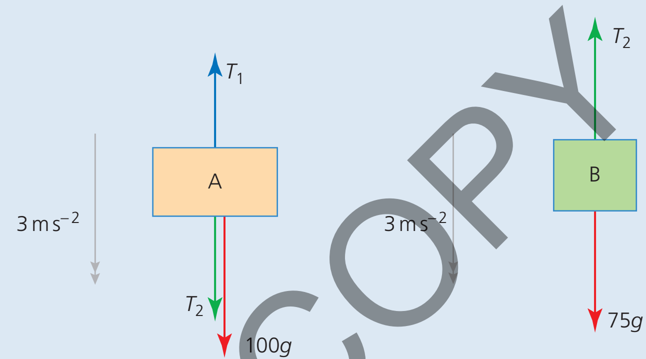


Figure 3.13 Acceleration downwards.

(ii) Box A: The resultant downwards force is $T_2 + 100g - T_1$ so the equation of motion is

$$T_2 + 100g - T_1 = 300 \quad ①$$

← Since $100a = 100 \times 3 = 300$

Box B: The resultant downwards force is $75g - T_2$ so the equation of motion is

$$75g - T_2 = 225 \quad ②$$

← Since $75a = 75 \times 3 = 225$

(iii) From ②: $T_2 = 75 \times 9.8 - 225$
$= 510\,\text{N}$

Substituting in ①: $T_1 = 510 + 100 \times 9.8 - 300$
$= 1190\,\text{N}$

The tension in the blue wire linking A to the parachute is 1365 N.
The tension in the green wire linking A to B is 510 N.

Note

T_1 is the tension in the blue wire linking A to the parachute. T_2 is the tension in the green wire linking A to B.
$T_1 \neq T_2$

Discussion point

Is the level of accuracy given in these answers justified?

Exercise 3.1

① Find the accelerations produced when a force of 100 N acts on an object

(i) of mass 15 kg

(ii) of mass 10 g

(ii) of mass 1 tonne.

② A bullet of mass 20 g is fired into a wall with a velocity of $400\,\text{m}\,\text{s}^{-1}$. The bullet penetrates the wall to a depth of 10 cm. Find the resistance of the wall, assuming it to be uniform.

③ A car of mass 1200 kg is travelling along a straight level road.

(i) Calculate the acceleration of the car when a resultant force of 2400 N acts on it in the direction of its motion. How long does it take the car to increase its speed from $4\,\mathrm{m\,s^{-1}}$ to $12\,\mathrm{m\,s^{-1}}$?

The car has an acceleration of $1.2\,\mathrm{m\,s^{-2}}$ when there is a driving force of 2400 N.

(ii) Find the resistance to motion of the car.

④ A load of mass 5 kg is held on the end of a string. Calculate the tension in the string when

(i) the load is raised with an acceleration of $2.5\,\mathrm{m\,s^{-2}}$

(ii) the load is lowered with an acceleration of $2.5\,\mathrm{m\,s^{-2}}$

(iii) the load is raised with a constant speed of $2\,\mathrm{m\,s^{-1}}$

(iv) the load is raised with a deceleration of $2.5\,\mathrm{m\,s^{-2}}$.

⑤ A block A of mass 10 kg is connected to a block B of mass 5 kg by a light inextensible string passing over a smooth fixed pulley. The blocks are released from rest with A 0.3 m above ground level, as shown in Figure 3.14.

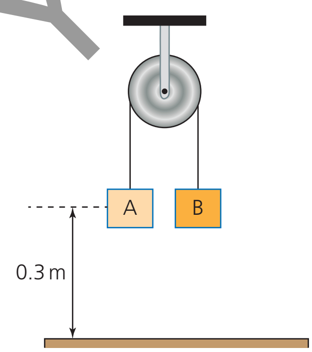


Figure 3.14

(i) Find the acceleration of the system and the tension in the string.

(ii) Find the speed of the masses when A hits the ground.

(iii) How far does B rise after A hits the floor and the string becomes slack?

⑥ A block A of mass 10 kg is lying on a smooth horizontal table. Light inextensible strings connect A to particle B of mass 6 kg and particle C of mass 4 kg, which hang freely over smooth pulleys at the edge of the table.

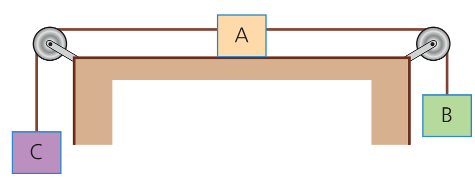


Figure 3.15

(i) Draw force diagrams to show the forces acting on each mass.

(ii) Write down separate equations of motion for A, B and C.

(iii) Find the acceleration of the system and the tensions in the strings.

⑦ A truck of mass 1250 kg is towing a trailer of mass 350 kg along a horizontal straight road. The engine of the truck produces a driving force of 2500 N. The truck is subjected to a resistance of 250 N and the trailer to a resistance of 300 N.

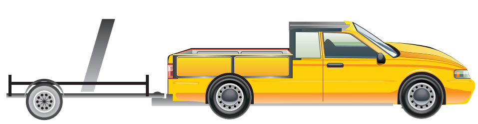


Figure 3.16

(i) Show, in separate diagrams, the horizontal forces acting on the truck and the trailer.

(ii) Find the acceleration of the truck and trailer.

(iii) Find the tension in the coupling between the truck and the trailer.

⑧ A train consists of a locomotive and five trucks with masses and resistances to motion as shown in Figure 3.17. The engine provides a driving force of 29 000 N. All the couplings are light, rigid and horizontal.

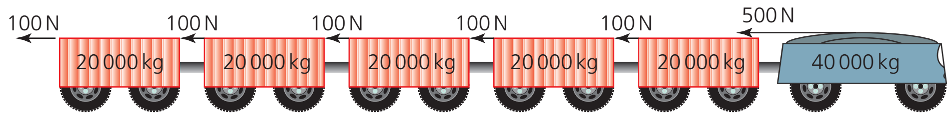


Figure 3.17

(i) Show that the acceleration of the train is $0.2\,\mathrm{m\,s^{-2}}$.

(ii) Find the force in the coupling between the last two trucks.

With the driving force removed, brakes are applied, so adding additional resistances of 3000 N to the locomotive and 2000 N to each truck.

(iii) Find the new acceleration of the train.

(iv) Find the force in the coupling between the last two trucks.

⑨ Block A of mass 2 kg is connected to a light scale pan by a light inextensible string which passes over a smooth fixed pulley.

The scale pan holds two blocks, B and C, of masses 0.5 and 1 kg, as shown in Figure 3.18.

(i) Draw diagrams showing all the forces acting on each of the three particles.

(ii) Write down equations of motion for each of A, B and C.

(iii) Find the acceleration of the system, the tension in the string, the reaction force between B and C and the reaction between C and the scale pan.

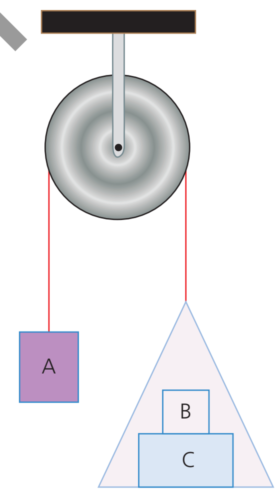


Figure 3.18

2 Working in vectors

A force is a physical quantity that causes a change of motion. A force can start the motion of an object, stop its motion, make it move faster or slower, or change the direction of its motion. By its very nature, a force is a vector quantity just like displacement, velocity or acceleration. It has magnitude and direction, unlike scalar quantities such as distance, speed, mass or time, which have magnitude only.

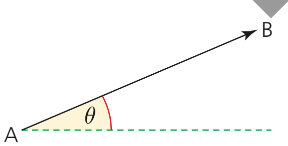


Figure 3.19

Notation and representation

A vector can be represented by a directed line segment in a diagram.

In writing, $\overrightarrow{AB}$ represents the vector with magnitude. The magnitude can also be written as $|\overrightarrow{AB}|$. The direction is given by the angle θ which AB makes with a fixed direction, often the horizontal.

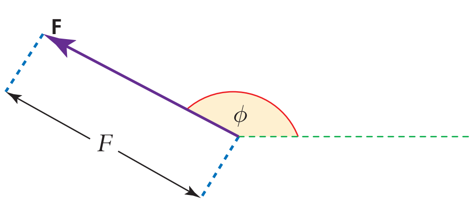


Figure 3.20

In Figure 3.20, $\mathbf{F}$ is a vector with magnitude $F = |\mathbf{F}|$ and direction ϕ.

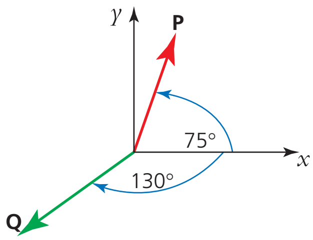


Figure 3.22

Vectors are often written using lowercase letters, like **a**, **b** and **c**. It is common to use **a** for the vector $\overrightarrow{OA}$ where O is the origin.

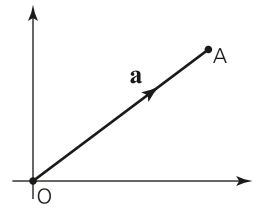


Figure 3.21

To determine the direction of a vector in the *xy* plane, a mathematical convention is used. Starting from the *x*-axis, angles measured anticlockwise are positive and angles in a clockwise direction are negative.

In the example shown in Figure 3.22, **P** has direction +75° and **Q** has direction –130°.

Adding vectors

One way to add vectors is to draw them one after another, i.e. where one finishes the next one starts.

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

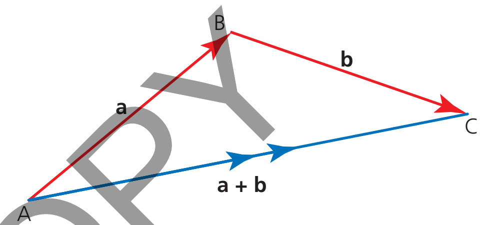


Figure 3.23

Alternatively, you can make **a** and **b** start at the same place and take the diagonal of the ensuing parallelogram.

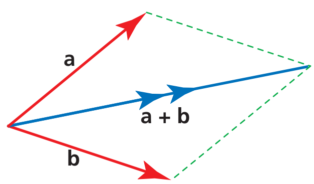


Figure 3.24

This gives the same result, because opposite sides of a parallelogram are equal and in the same direction, so that **b** is repeated at the top right of the parallelogram in Figure 3.24.

Resultant forces

The resultant of a number of forces is equal to the sum of these forces. As each force is a vector, the resultant is a vector starting at the start point of the first force and ending at the end point of the last force.

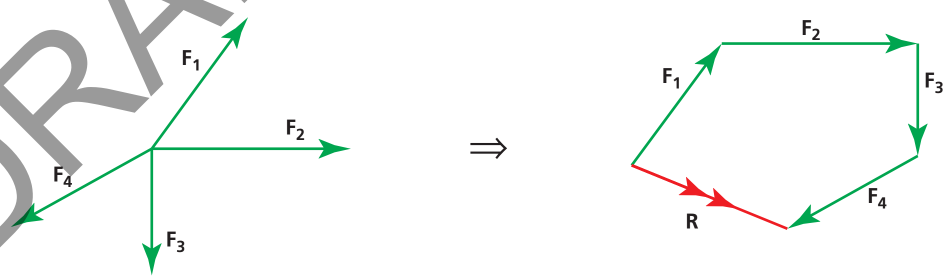


Figure 3.25

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4$$

The resultant **R** of the four forces ($\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ and $\mathbf{F}_4$) can be found by drawing consecutive lines representing the vectors. The line which completes the polygon is the resultant.

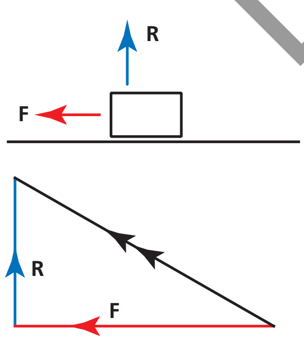


Figure 3.26

You have met the contact forces of normal reaction **R** and frictional force **F**. The resultant of them is the *total contact force* with magnitude $\sqrt{F^2+R^2}$.

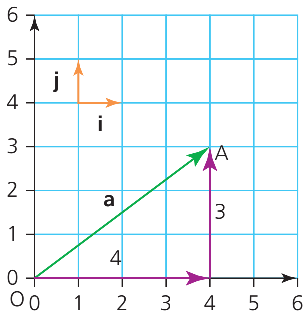


Figure 3.27

Components of a vector

Finding components is the reverse process of adding two vectors. It involves splitting a vector into two perpendicular components.

The result is often described using unit vectors **i** and **j** along the x and y axes respectively. The vector **a** in Figure 3.27 may be written as **a** = 4**i** + 3**j**.

Alternatively, **a** can be written as the *column vector* $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

In general if **i** and **j** are unit vectors along the x and y directions respectively then **a** can be written in terms of components as $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j}$ or in column vector form

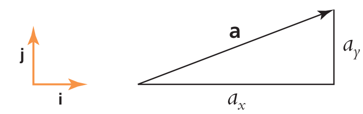


Figure 3.28

$$\mathbf{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$$

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$$

Example 3.5

The four vectors **a**, **b**, **c** and **d** are shown in the diagram.

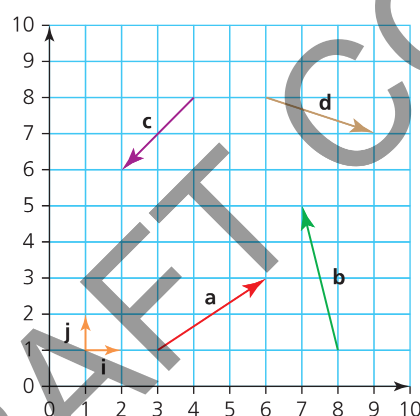


Figure 3.29

(i) Write them in component form and as column vectors.

(ii) Draw a diagram to show the vectors 2**a**, –**b** and 2**a** – **b** and write these in both forms.

Solution

(i) $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\mathbf{b} = -\mathbf{i} + 4\mathbf{j} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$, $\mathbf{c} = -2\mathbf{i} - 2\mathbf{j} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$,

$\mathbf{d} = 3\mathbf{i} - \mathbf{j} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

(ii)

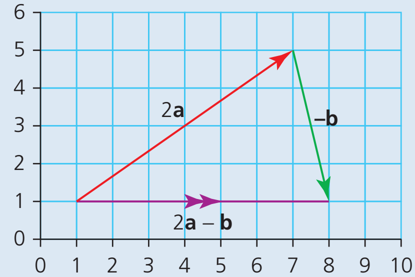


Figure 3.30

$$2\mathbf{a} = 2(3\mathbf{i} + 2\mathbf{j}) = 6\mathbf{i} + 4\mathbf{j} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

$$-\mathbf{b} = -(-\mathbf{i} + 4\mathbf{j}) = \mathbf{i} - 4\mathbf{j} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

$$2\mathbf{a} - \mathbf{b} = 2\mathbf{a} + (-\mathbf{b}) = 6\mathbf{i} + 4\mathbf{j} + \mathbf{i} - 4\mathbf{j} = 7\mathbf{i} = \begin{pmatrix} 7 \\ 0 \end{pmatrix}$$

Position vectors

To specify the position of an object, you define its displacement relative to a fixed origin. **a** and **b** are usually used to define the position vectors of A and B.

$\mathbf{a} = \overrightarrow{OA}, \mathbf{b} = \overrightarrow{OB}$
$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$
$\overrightarrow{AB} = -\mathbf{a} + \mathbf{b}$
$= \mathbf{b} - \mathbf{a}$

The vector between two points

The displacement AB can be replaced by the displacement from A to O followed by that from O to B.

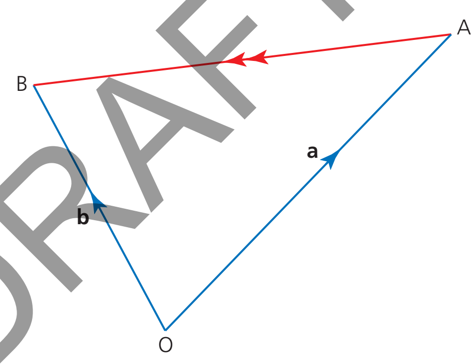


Figure 3.31

Any displacement vector AB can be written in terms of the position vectors of its two end points.

Given a vector $a = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$ at an angle of θ with the x axis, then

- If a_x is positive then $\theta = \arctan\begin{pmatrix} a_y \\ a_x \end{pmatrix}$
- If a_x is negative and a_y is positive then $\theta = \arctan\begin{pmatrix} a_y \\ a_x \end{pmatrix} + 180$ (or in radians this is $\begin{pmatrix} a_y \\ a_x \end{pmatrix} + \pi$)
- If a_x is negative and a_y is negative then $\theta = \arctan\begin{pmatrix} a_y \\ a_x \end{pmatrix} - 180$ (or in radians this is $\begin{pmatrix} a_y \\ a_x \end{pmatrix} - \pi$).

The magnitude and direction of vectors written in component form

The magnitude of a vector is just its length and can be found by using Pythagoras' theorem.

$$a = \sqrt{a_x^{\,2} + a_y^{\,2}}$$

The direction is related to the angle the vector makes with the positive x-axis.

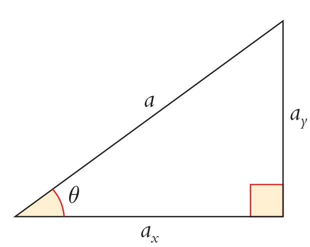


Figure 3.32

$$\theta = \arctan\left(\frac{a_y}{a_x}\right)$$

or $\theta = \arctan\left(\frac{a_y}{a_x}\right) \pm 180°$, depending on which quadrant a is in.

Example 3.6

Find the magnitude and direction of each of the four vectors

$\mathbf{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$, $\mathbf{c} = -2\mathbf{i} - 2\mathbf{j}$ and $\mathbf{d} = 3\mathbf{i} - \mathbf{j}$.

Solution

$\mathbf{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

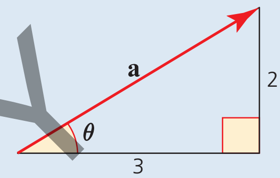


Figure 3.33

Magnitude $|\mathbf{a}| = \sqrt{3^2 + 2^2} = \sqrt{13} = 3.61$

Direction $\theta = \arctan\left(\frac{2}{3}\right) = 33.7°$

$\mathbf{b} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$

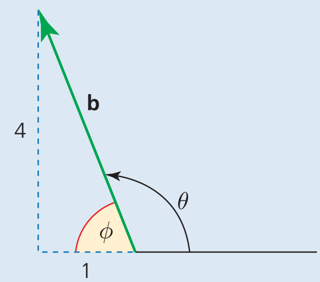


Figure 3.34

Magnitude $|\mathbf{b}| = \sqrt{(-1)^2 + 4^2} = \sqrt{17} = 4.12$

$\tan\phi = \frac{4}{1} \Rightarrow \phi = \arctan 4 = 76.0°$

Direction $\theta = 180 - \phi = 104°$

$\mathbf{c} = -2\mathbf{i} - 2\mathbf{j}$

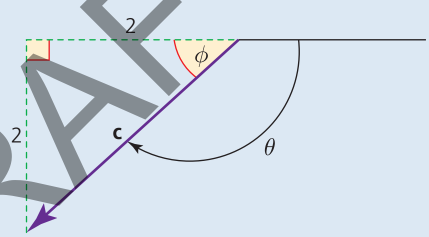


Figure 3.35

Magnitude $|\mathbf{c}| = \sqrt{(-2)^2 + (-2)^2} = \sqrt{8} = 2.83$

$\tan\phi = \frac{2}{2} = 1 \Rightarrow \phi = \arctan 1 = 45°$

Direction $\theta = -(180° - 45°) = -135°$

$\mathbf{d} = 3\mathbf{i} - \mathbf{j}$

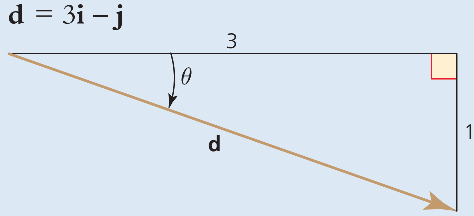


Figure 3.36

Magnitude $|\mathbf{d}| = \sqrt{3^2 + (-1)^2} = \sqrt{10} = 3.16$

Direction $\theta = -\arctan\left(\frac{1}{3}\right) = -18.4°$

Finding unit vectors along given directions

If $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j}$, the magnitude of $\mathbf{a}$ is $|\mathbf{a}| = \sqrt{a_x^{\ 2} + a_y^{\ 2}}$. A *unit vector* along $\mathbf{a}$ (denoted by $\hat{\mathbf{a}}$) has magnitude 1. $\hat{\mathbf{a}}$ is parallel to $\mathbf{a}$ but has a magnitude which is scaled by the factor $\frac{1}{|\mathbf{a}|}$.

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{a_x}{\sqrt{a_x^{\ 2} + a_y^{\ 2}}}\mathbf{i} + \frac{a_y}{\sqrt{a_x^{\ 2} + a_y^{\ 2}}}\mathbf{j}$$

Example 3.7

(i) Find a unit vector along $\mathbf{a} = 3.5\mathbf{i} - 12\mathbf{j}$

(ii) Find a vector along $\mathbf{a}$ which is 25 units long.

Solution

(i) The magnitude of $\mathbf{a}$ is $|\mathbf{a}| = \sqrt{3.5^2 + (-12)^2} = \sqrt{12.25 + 144}$

$= \sqrt{156.25} = 12.5$

A unit vector along $\mathbf{a}$ is $\hat{\mathbf{a}} = \frac{3.5}{12.5}\mathbf{i} - \frac{12}{12.5}\mathbf{j} = 0.28\mathbf{i} - 0.96\mathbf{j}$

(ii) $\hat{\mathbf{a}}$ has magnitude 1, so you are looking for a vector along $\mathbf{a}$ which is 25 units long, i.e. $25\hat{\mathbf{a}} = 25(0.28\mathbf{i} - 0.96\mathbf{j}) = 7\mathbf{i} - 24\mathbf{j}$

Resolving a force into components in two perpendicular directions

Draw the vector **F**, magnitude *F*, making an angle θ with the *x*-axis, taken as the ***i*** direction. Make up the right-angled triangle with **F** along the hypotenuse and the *x* and *y* components along the other two sides. These are then evaluated using trigonometry. This process is called *resolving* F into its *x* and *y* components.

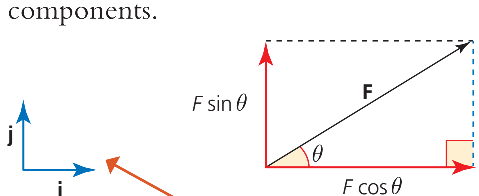


Figure 3.37

$$F = F\cos\theta\mathbf{i} + F\sin\theta\mathbf{j} = \begin{pmatrix} F\cos\theta \\ F\sin\theta \end{pmatrix}$$

Example 3.8

Resolve a weight W N in two directions which are along and at right angles to a slope making an angle θ with the horizontal.

Solution

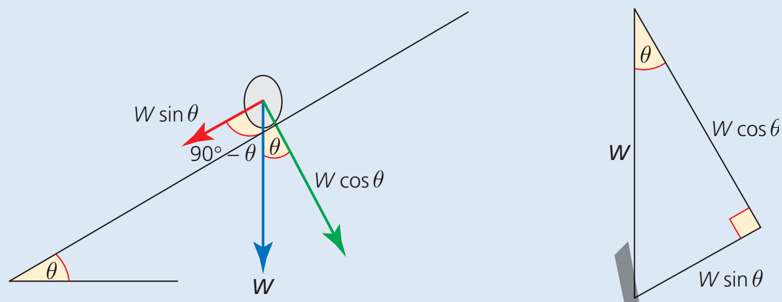


Figure 3.38

W is the hypotenuse of the right-angled triangle. The components are along the other two sides. The component parallel to the slope is $W \cos(90° - \theta) = W \sin \theta$.

The component perpendicular to the slope is $W \cos \theta$.

Example 3.9

Two forces **P** and **Q** have magnitudes 10 N and 15 N in the directions shown in Figure 3.39.

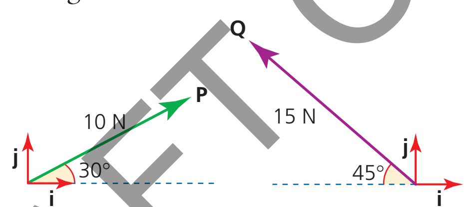


Figure 3.39

Find the magnitude and direction of the resultant force **P** + **Q**.

Solution

Note
P = 10 cos 30° **i** + 10 sin 30° **j**
= 8.66 **i** + 5 **j**

Note
Q = –15 cos 45° **i** + 15 sin 45° **j**
= –10.61 **i** + 10.61 **j**

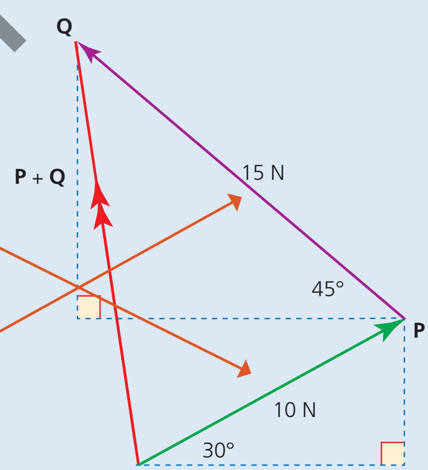


Figure 3.40

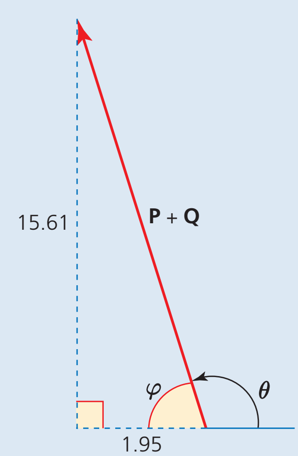


Figure 3.41

The resultant is $\mathbf{P} + \mathbf{Q} = (8.66\mathbf{i} + 5\mathbf{j}) + (-10.61\mathbf{i} + 10.61\mathbf{j})$

$= -1.95\mathbf{i} + 15.61\mathbf{j}$

It is shown in Figure 3.41.

The magnitude of the resultant is $|\mathbf{P}+\mathbf{Q}| = \sqrt{(-1.95)^2 + 15.61^2} = 15.73$

The direction of the resultant:

$$\tan\varphi = \frac{15.61}{1.95} \quad \Rightarrow \quad \varphi = \arctan(8.01) = 82.9°$$
$$\theta = 180° - 82.9° = 97.1°$$

The resultant force **P** + **Q** has magnitude 15.7 and direction 97.1° relative to the positive x-axis.

Exercise 3.2

① Four vectors are given in component form by $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$, $\mathbf{b} = 6\mathbf{i} - 7\mathbf{j}$, $\mathbf{c} = -2\mathbf{i} + 5\mathbf{j}$ and $\mathbf{d} = -5\mathbf{i} - 3\mathbf{j}$.

Find the vectors

(i) $\mathbf{a} + \mathbf{b}$ (ii) $\mathbf{b} + \mathbf{c}$ (iii) $\mathbf{c} + \mathbf{d}$

(iv) $\mathbf{a} + \mathbf{b} + \mathbf{d}$ (v) $\mathbf{a} - \mathbf{b}$ (vi) $\mathbf{d} - \mathbf{b} + \mathbf{c}$

② A, B, C are the points (1, 2), (5, 1) and (7, 8).

(i) Write down in terms of **i** and **j** the position vectors of these three points.

(ii) Find the component form of the displacements $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$.

(iii) Draw a diagram to show the position vectors of A, B and C and your answers to part (ii).

③ Three vectors **a**, **b** and **c** are given by $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$.

R is the endpoint of the displacement $2\mathbf{a} - 3\mathbf{b} + \mathbf{c}$ and (1, –2) is the starting point. What is the position vector of R?

④ Find the magnitude and direction of the following vectors.

(i) $12\mathbf{i} - 5\mathbf{j}$ (ii) $7\mathbf{i} + 24\mathbf{j}$ (iii) $-\mathbf{i} + \mathbf{j}$

(iv) $3\mathbf{i} + 4\mathbf{j}$ (v) $2\mathbf{i} - 3\mathbf{j}$ (vi) $-\mathbf{i} - 2\mathbf{j}$

⑤ Write down the following vectors in component form in terms of **i** and **j**.

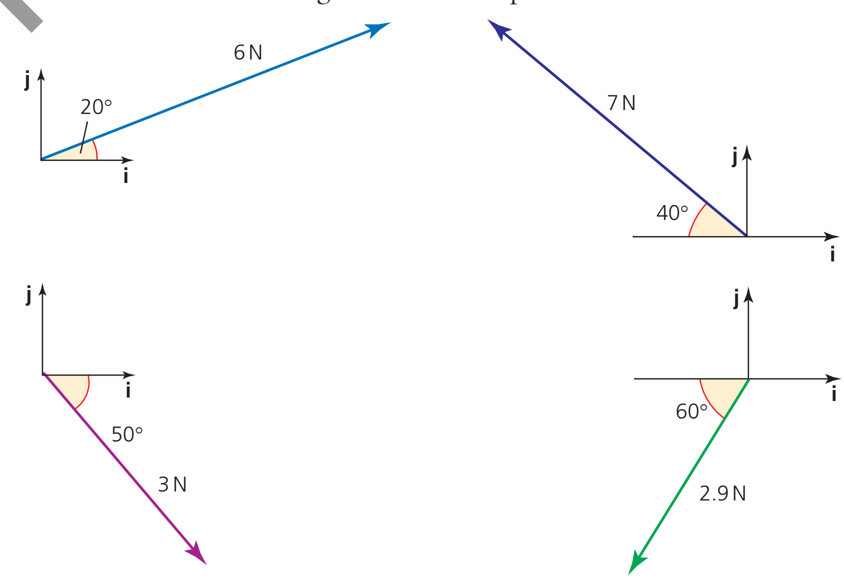


Figure 3.42

⑥ (i) Find a unit vector in the direction of $\begin{pmatrix} 10 \\ 24 \end{pmatrix}$.

A force **F** acts in the direction of $\begin{pmatrix} 10 \\ 24 \end{pmatrix}$ and has magnitude 39 N.

(ii) Use your answer to part (i) to write **F** in component form.

⑦ Find the vector with magnitude 8.2 that is parallel to the vector 40**i** − 9**j**.

⑧ Write down each of the following vectors in terms of **i** and **j**. Find the resultant of each set of vectors in terms of **i** and **j**.

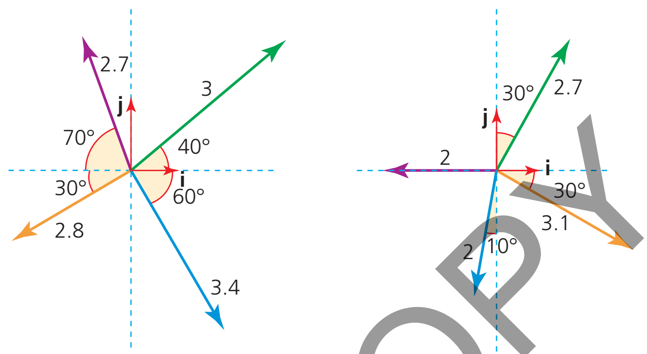


Figure 3.43

⑨ The displacement of B from A is $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$. The displacement of C from A is $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$. The displacement of D from A is $\begin{pmatrix} 0 \\ -4 \end{pmatrix}$.

Draw a diagram showing the relative position of A, B, C and D. Find

(i) $\overrightarrow{DB}$ (ii) $\overrightarrow{DC}$ (iii) $\overrightarrow{CB}$ (iv) $\overrightarrow{BC}$

⑩ Three vectors **a**, **b** and **c** are represented by the sides of a triangle ABC, as shown in Figure 3.44.

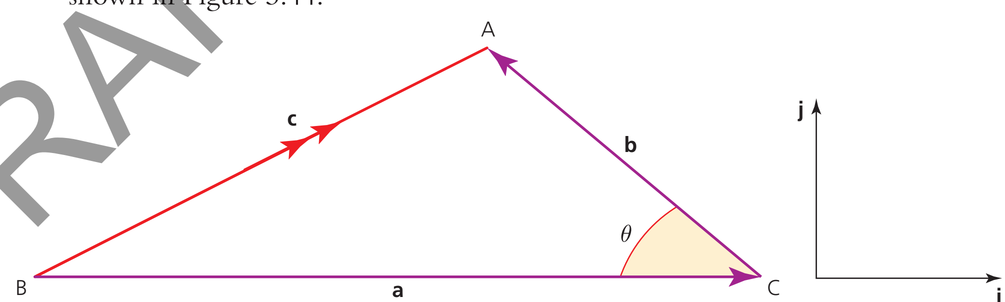


Figure 3.44

The angle C is θ and |**a**|, |**b**| and |**c**| are a, b and c. Answer each part in terms of θ, a, b and c.

(i) Write **a** and **b** in terms of **i** and **j**.

(ii) Find **a** + **b** and hence |**a** + **b**|^2.

(iii) Use your answer to part (ii) to express c^2 in terms of a, b and θ.

3 Forces in equilibrium

When forces are in equilibrium their vector sum is zero and the sum of the resolved parts in *any* direction is zero.

Example 3.10

A brick of mass 5 kg is at rest on a rough plane inclined at an angle of 35° to the horizontal. Find the frictional force F N, and the normal reaction R N of the plane on the brick.

Solution

The diagram shows the forces acting on the brick.

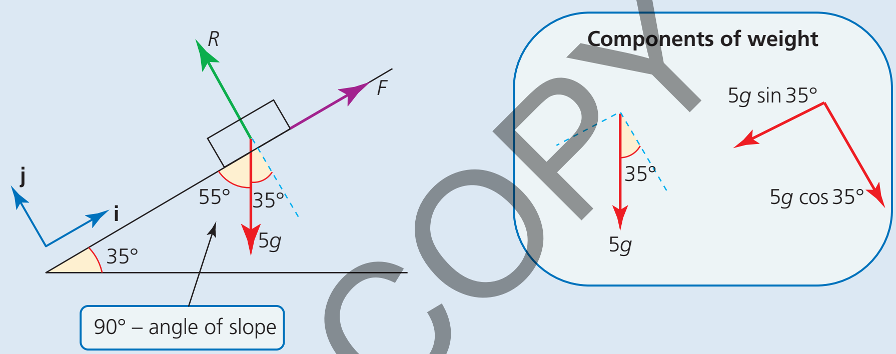


Figure 3.45

Take unit vectors **i** and **j** parallel and perpendicular to the plane, as shown.

Since the brick is in equilibrium, the resultant of the three forces acting on it is zero.

$5g = 49$

Resolving in the **i** direction: $F - 49 \sin 35° = 0$ ①

$$F = 28.10 \ldots$$

Resolving in the **j** direction: $R - 49 \cos 35° = 0$ ②

$$R = 40.13 \ldots$$

Written in vector form this is equivalent to

$$F\mathbf{i} + R\mathbf{j} - 49 \sin 35°\mathbf{i} - 49 \cos 35°\mathbf{j} = 0$$

or, alternatively,

$$\begin{pmatrix} F \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ R \end{pmatrix} + \begin{pmatrix} -49 \sin 35° \\ -49 \cos 35° \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Notice that both of these vector equations lead to the equations ① and ② above.

The triangle of forces

When there are only three (non-parallel) forces acting and they are in equilibrium, the polygon of forces becomes a closed triangle, as shown for the brick on the plane in Figures 3.46 and 3.47.

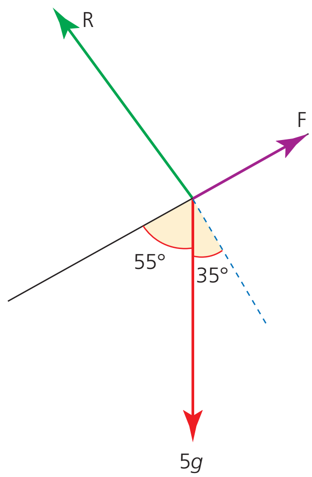


Figure 3.46

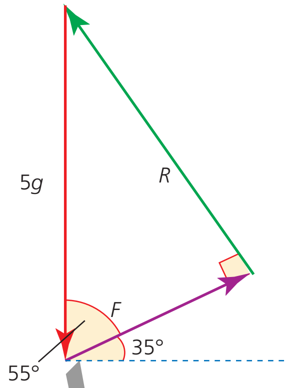


Figure 3.47

Note

The triangle is closed because the resultant is zero.

When a body is in equilibrium under the action of three non-parallel forces, then

(i) the forces can be represented in magnitude and direction by the sides of a triangle

(ii) the lines of action of the forces pass through the same point. They are concurrent.

Then $$\frac{F}{5g} = \cos 55°$$

$$F = 49\cos 55° = 28.1\,\text{N}$$

And similarly $$R = 49\sin 55° = 40.1\,\text{N}$$

This is an example of the theorem known as the *triangle of forces.*

When more than three forces are in equilibrium, the first statement still holds but the triangle is then a polygon. The second statement is not necessarily true.

The next example illustrates two methods for solving problems involving forces in equilibrium. With experience, you will find it easier to judge which method is more suitable for a particular problem.

Example 3.11

A sign of mass 10 kg is to be suspended by two strings arranged as shown in Figure 3.48. Find the tension in each string.

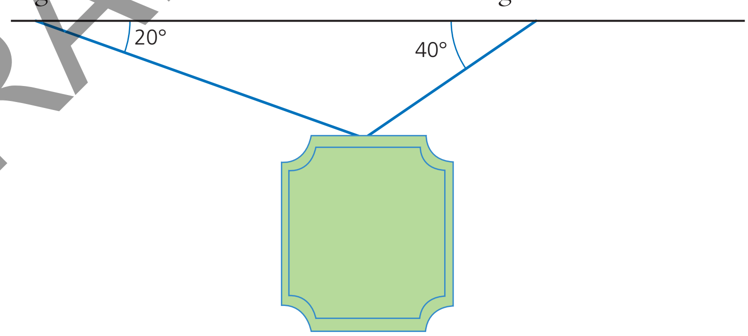


Figure 3.48

Solution

The force diagram for this situation is given below.

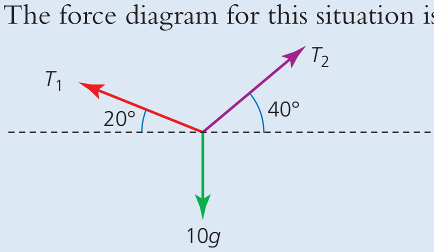


Figure 3.49

Method 1: Resolving forces

Vertically (↑): $T_1 \sin 20° + T_2 \sin 40° - 10g = 0$ ①

$0.342\ldots T_1 + 0.642\ldots T_2 = 98$

Horizontally (→): $-T_1 \cos 20° + T_2 \cos 40° = 0$ ②

$-0.939\ldots T_1 + 0.766\ldots T_2 = 0$

The set of simultaneous equations is solved in the usual way. Whether you are using the equation solver on your calculator or working it out on paper, it is important that you keep as much accuracy as possible by substituting for the different sines and cosines only at the very end of the calculation.

Multiply ① by cos 20° and then add ② × sin 20° to give

You may recognise that this is the compound angle form for sin (40° + 20°) and so is the same as sin 60°.

$T_2(\sin 40° \cos 20° + \cos 40° \sin 20°) = 98 \cos 20°$

$$T_2 = \frac{98 \cos 20°}{\sin 40° \cos 20° + \cos 40° \sin 20°}$$

$$T_2 = 106.33\ldots$$

Substituting back in ② now gives

$$T_1 = \frac{T_2 \cos 40°}{\cos 20°} = 86.68\ldots$$

The tensions in the strings are 87 N and 106 N.

Method 2: Triangle of forces

Since the three forces are in equilibrium they can be represented by the sides of a triangle taken in order.

Discussion point

In what order would you draw the three lines in this triangle of forces diagram?

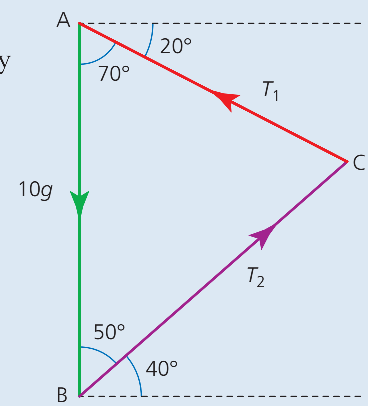


Figure 3.50

You can estimate the tensions by measurements. This will tell you that $T_1 \approx 87$ and $T_2 \approx 106$ in newtons.

Alternatively, you can use the sine rule to calculate T_1 and T_2 accurately.

In triangle ABC, $\widehat{CAB} = 70°$ and $\widehat{ABC} = 50°$, so $\widehat{BCA} = 60°$.

180° − 70° − 50° = 60°

So $$\frac{T_1}{\sin 50°} = \frac{T_2}{\sin 70°} = \frac{98}{\sin 60°}$$

giving $$T_1 = \frac{98 \sin 50°}{\sin 60°} \text{ and } T_2 = \frac{98 \sin 70°}{\sin 60°}$$

As before the tensions are found to be 87 N and 106 N.

Discussion point

Lami's theorem states that when three forces acting at a point as shown in the diagram are in equilibrium then

$$\frac{F_1}{\sin\alpha} = \frac{F_2}{\sin\beta} = \frac{F_3}{\sin\gamma}.$$

Sketch a triangle of forces and say how the angles in the triangle are related to α, β and γ. Hence explain why Lami's theorem is true.

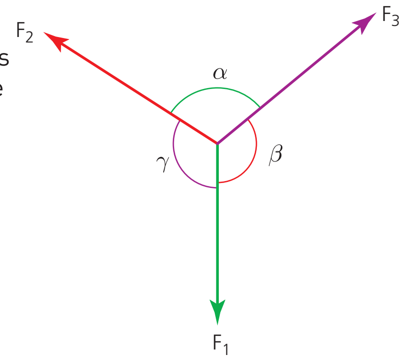


Figure 3.51

Example 3.12

Two husky dogs are pulling a sledge. They both exert forces of 60 N but at different angles to the line of the sledge, as shown in the diagram. The sledge is moving straight forwards.

Figure 3.52

(i) Resolve the two forces into components parallel and perpendicular to the line of the sledge.

(ii) Find the overall forward force from the dogs and the overall sideways force.

The resistance to motion is 20 N along the line of the sledge and up to 400 N perpendicular to it.

(iii) Find the magnitude and direction of the overall horizontal force on the sledge.

(iv) How much force is lost due to the dogs not pulling straight forwards?

Solution

(i) Taking unit vectors **i** along the line of the sledge and **j** perpendicular to the line of the sledge.

The forces exerted by the two dogs are

$$60\cos 15°\mathbf{i} + 60\sin 15°\mathbf{j}$$

$$= 57.95\ldots\mathbf{i} + 15.52\ldots\mathbf{j}$$

and $$60\cos 10°\mathbf{i} - 60\sin 10°\mathbf{j}$$

$$= 59.08\ldots\mathbf{i} - 10.41\ldots\mathbf{j}$$

(ii) The overall forward force is equal to

$$60\cos 15° + 60\cos 10° = 117.04\ldots = 117\,\text{N}$$

The overall sideways force is equal to

$$60\sin 15° - 60\sin 10° = 5.11\ldots = 5.1\,\text{N}$$

(iii) The sideways force is cancelled by the resistance force opposing it.

The forward force is reduced by an amount 20 N from the resistance to motion.

So that the overall forward force is 97 N and the overall sideways force is 0.

The magnitude of the overall force on the sledge is thus 97 N in the direction of motion.

(iv) If the dogs were pulling straight, the overall force on the sledge would be 100 N, so the amount of force lost due to the dogs not pulling straight is thus 3 N.

100 – 97.04...

60 + 60 – 20 (60 N from each dog less 20 N from the resistance)

Example 3.13

The tension in the string is the same on either side of the pulley. This is a consequence of there being no friction between the pulley and the axle, and the pulley having no mass. The string does not slide over the pulley but moves with it.

Two blocks P and Q have masses m kg and M kg. They are connected by a light inextensible string that runs over a smooth light pulley.
The block P rests on a plane where the coefficient of friction is μ. The block Q hangs freely, as in the diagram.

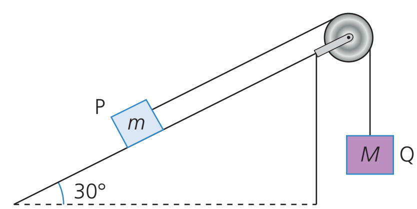


Figure 3.53

When $M = 4$, the block P is just about to slide up the slope. When $M = 1$, the block P is just about to slide down the slope. Find μ and m.

Solution

When P is about to slide up the slope, the situation is as shown in Figure 3.54 where the tension in the string is T N and the normal reaction of the plane on P is N N.

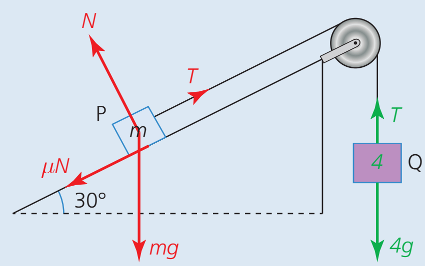


Figure 3.54

The equilibrium of Q $\Rightarrow T = 4g$.

Resolving up the plane for P $\Rightarrow T = \mu N + mg\sin 30°$

Resolving perpendicular to the plane for P $\Rightarrow N = mg\cos 30°$.

Thus $4g = \mu mg\frac{\sqrt{3}}{2} + mg\frac{1}{2} \Rightarrow \mu = \frac{8-m}{\sqrt{3}m}$.

Figure 3.55 shows the situation when P is about to slide down the slope. The tension in the string is now T' N.

The block Q $\Rightarrow T' = g$.

Resolving up the plane for P $\Rightarrow T' + \mu N' = mg\sin 30°$.

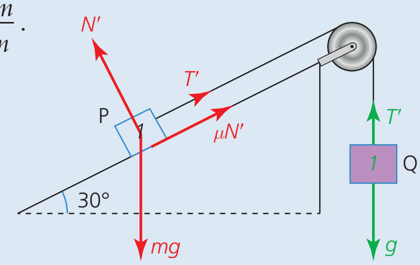


Figure 3.55

Resolving perpendicular to the plane for P $\Rightarrow N' = mg\cos 30°$.

Thus $g + \mu mg\frac{\sqrt{3}}{2} = mg\frac{1}{2} \Rightarrow \mu = \frac{m-2}{\sqrt{3}m}$.

Equating $4g = \mu mg\frac{\sqrt{3}}{2} + mg\frac{1}{2} \Rightarrow \mu = \frac{8-m}{\sqrt{3}m}$ and $g + \mu mg\frac{\sqrt{3}}{2} = mg\frac{1}{2}$

$\Rightarrow \mu = \frac{m-2}{\sqrt{3}m}$

$\Rightarrow m - 2 = 8 - m,$

and so $m = 5 \Rightarrow \mu = \frac{\sqrt{3}}{5} = 0.35$ (2 s.f.).

Exercise 3.3

① The following sets of forces are in equilibrium. Find the value of p and q in each case.

(i) $\begin{pmatrix}24\\18\end{pmatrix}$N, $\begin{pmatrix}25\\60\end{pmatrix}$N and $\begin{pmatrix}p\\q\end{pmatrix}$N

(ii) $\begin{pmatrix}p\\-2\end{pmatrix}$N, $\begin{pmatrix}-3\\4\end{pmatrix}$N and $\begin{pmatrix}2\\-q\end{pmatrix}$N

(iii) $\begin{pmatrix}2p\\5\end{pmatrix}$N, $\begin{pmatrix}q\\4p\end{pmatrix}$N, $\begin{pmatrix}p\\-3\end{pmatrix}$N and $\begin{pmatrix}5\\-q\end{pmatrix}$N.

② A brick of mass 2 kg is resting on a rough plane inclined at 40° to the horizontal.

(i) Draw a diagram showing all the forces acting on the brick.

(ii) Find the normal reaction of the plane on the brick.

(iii) Find the frictional force acting on the brick.

③ A particle is in equilibrium under the three forces shown in Figure 3.56. Find the magnitude of the force F and the angle θ.

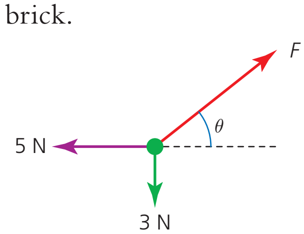


Figure 3.56

④ A box of mass 10 kg is at rest on a horizontal floor.

(i) Find the value of the normal reaction of the floor on the box.

The box remains at rest on the floor when a force of 30 N is applied to it at an angle of 25° to the upward vertical as shown in Figure 3.57.

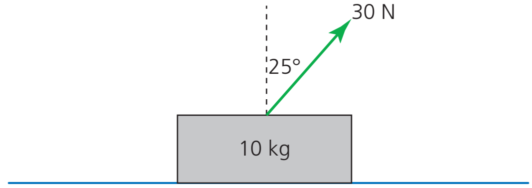


Figure 3.57

(i) Draw a diagram showing all the forces acting on the box.

(ii) Calculate the new value of the normal reaction of the floor on the box and also the frictional force.

5 A block of weight 100 N is on a rough plane that is inclined at 30° to the horizontal. The block is in equilibrium with a force of 35 N acting on it, in the direction of the plane as shown in Figure 3.58.

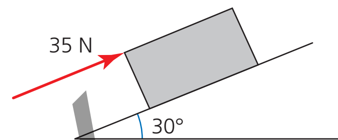


Figure 3.58

Calculate the frictional force acting on the block.

6 A crate of mass 10 kg is being pulled across rough horizontal ground by a rope making an angle θ with the horizontal. The tension in the rope is 60 N and the frictional force between the crate and the ground is 35 N. The crate is in equilibrium.

(i) Draw a labelled diagram showing all the forces acting on the crate.

(ii) Find the angle θ.

(iii) Find the normal reaction between the floor and the crate.

7 Each of three light strings has a block attached to one of its ends. The other ends of the strings are tied together at a point A. The strings are in equilibrium with two of them passing over fixed smooth pulleys and with the blocks hanging freely.

The weights of the blocks, and the angles between the sloping parts of the strings and the vertical, are as shown in Figure 3.59. Find the values of W_1 and W_2.

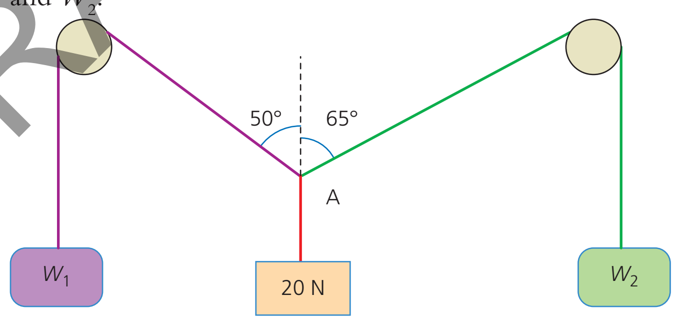


Figure 3.59

8 A block of mass 10 kg rests in equilibrium on a smooth plane inclined at 30° to the horizontal. It is held by a light string making an angle of 15° with the line of greatest slope of the plane.

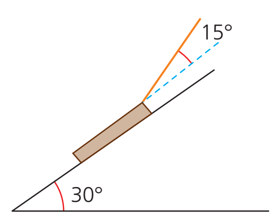


Figure 3.60

(i) Draw a labelled diagram showing all the forces acting on the block.

(ii) Find the tension in the string and the normal reaction of the plane on the block.

9. A particle A of mass 3 kg is at rest in equilibrium on horizontal rough ground. A is attached to two light, inextensible strings making angles of 20° and 50° with the vertical. The tensions in the two strings are 10 N and 20 N, as shown in Figure 3.61.

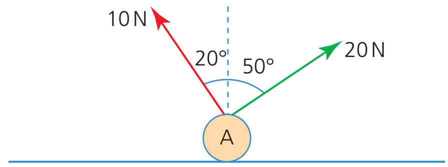


Figure 3.61

(i) Draw a diagram showing all the forces acting on A.

(ii) Find the normal reaction between the ground and A.

(iii) Find the magnitude of the frictional force, indicating the direction in which it is acting.

10. Four wires, all of them horizontal, are attached to the top of a telegraph pole as shown in this plan view. There is no overall force on the pole and tensions in the wires are as shown.

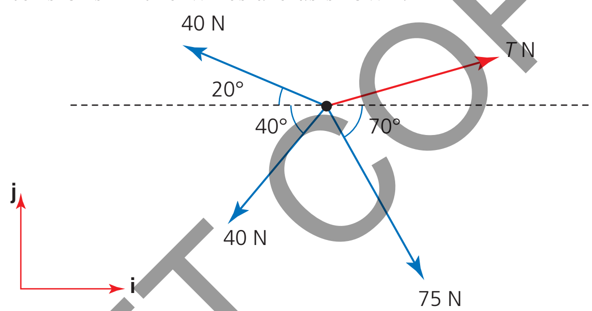


Figure 3.62

(i) Using perpendicular directions as shown in the diagram, show that the force of 75 N may be written as $(25.7\mathbf{i} - 70.5\mathbf{j})$ N (to 3 s.f.).

(ii) Find T in both component form and magnitude and direction form.

(iii) The force T is changed to $(40\mathbf{i} + 50\mathbf{j})$ N. Show that there is now a resultant force on the pole and find its magnitude and direction.

11. A ship is being towed by two tugs. They exert forces on the ship as indicated.

There is also a drag force on the ship.

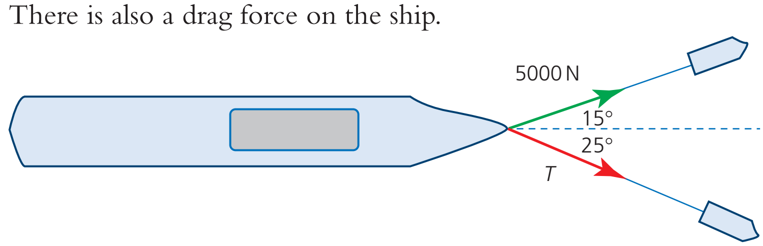


Figure 3.63

(i) Write down the components of the tensions in the towing cables along and perpendicular to the line of motion, l, of the ship.

(ii) There is no resultant force perpendicular to the line l. Find T.

(iii) The ship is travelling with constant velocity along the line l. Find the magnitude of the drag force acting on it.

⑫ The diagram shows a block of mass 10 kg on a rough inclined plane. The block is attached to a 7 kg weight by a light string which passes over a smooth pulley; it is on the point of sliding up the slope.

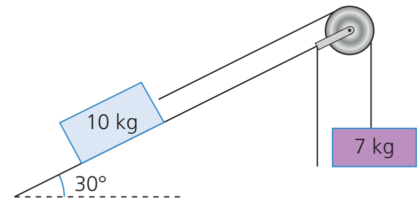


Figure 3.64

(i) Draw a diagram showing the forces acting on the block.

(ii) Resolve these forces into components parallel and perpendicular to the slope.

(iii) Find the force of resistance to the block's motion.

The 7 kg mass is replaced by one of mass m kg.

(iv) Find the value of m for which the block is on the point of sliding down the slope, assuming the resistance to motion is the same as before.

⑬ A block of mass 75 kg is in equilibrium on smooth horizontal ground with one end of a light string attached to its upper edge. The string passes over a smooth pulley, with a block of mass m kg attached at the other end.

The part of the string between the pulley and the block makes an angle of 65° with the horizontal. A horizontal force F is also acting on the block.

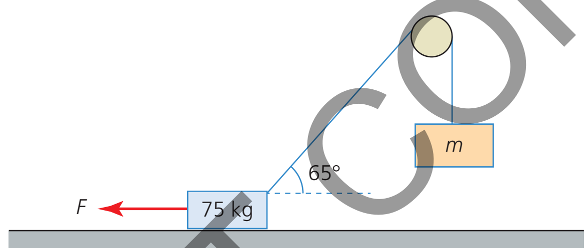


Figure 3.65

(i) Find a relationship between T, the tension in the string, and R, the normal reaction between the block and the ground.

The block is on the point of lifting off the ground.

(ii) Find T and m.

(iii) Find F.

4 Finding resultant forces

When forces are in equilibrium their resultant is zero, however forces are not always in equilibrium. The next example shows you how to find the resultant of forces that are not in equilibrium. You know from Newton's second law that the acceleration of the body will be in the same direction as the resultant force; remember that force and acceleration are both vector quantities.

Example 3.14

A sledge is being pulled up a smooth slope inclined at an angle of 15° to the horizontal by a rope which makes an angle of 30° with the slope. The mass of the sledge is 5 kg and the tension in the rope is 40 N.

(i) Draw a diagram to show the forces acting on the sledge.

(ii) Find the resultant of these forces.

(iii) Find the acceleration of the sledge.

Note

When the sledge is modelled as a particle, all the forces can be assumed to be acting at a point.

Note

There is no friction force because the slope is smooth.

Hint

Notice that although the sledge is moving up the slope this does not mean that the resultant force is up the slope. Its direction depends on the acceleration of the sledge which may be up or down the slope, or zero if the sledge is moving at constant speed.

Discussion point

Try resolving horizontally and vertically. You will obtain 2 equations in the two unknowns F and R. It is perfectly possible to solve these equations, but is quite a lot of work. How can you decide which directions will be easiest to work with?

Solution

(i) Figure 3.66 shows the force diagram.

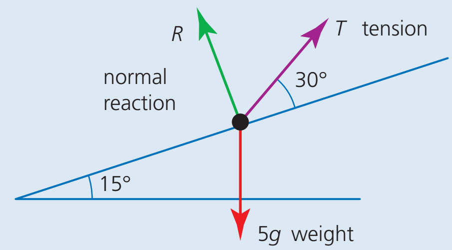


Figure 3.66

(ii) ***Method 1***

Resolve the forces into components parallel and perpendicular to the slope.

Components of the weight **Components of the tension**

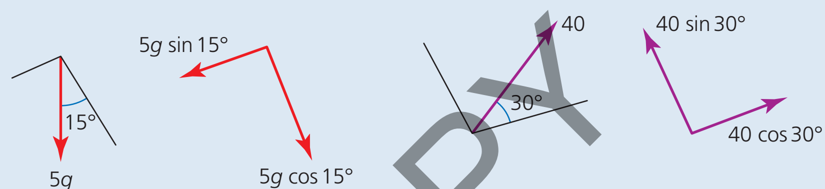


Figure 3.67

Resolve parallel to the slope: ↗

The resultant is $F = 40\cos 30° - 5g\sin 15°$
$= 21.959\ldots$

The force **R** is perpendicular to the slope so it has no component in this direction.

Resolve perpendicular to the slope: ↘

$$R + 40\sin 30° - 5g\cos 15° = 0$$
$$R = 5g\cos 15° - 40\sin 30° = 27.33$$

There is no resultant in this direction because the motion is parallel to the slope.

To 3 significant figures, the normal reaction is 27.3 N and the resultant is 22.0 N up the slope.

Method 2

Alternatively, you could have worked in column vectors as follows.

$$\begin{pmatrix} 0 \\ R \end{pmatrix} + \begin{pmatrix} 40\cos 30° \\ 40\sin 30° \end{pmatrix} + \begin{pmatrix} -5g\sin 15° \\ -5g\cos 15° \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}$$

Parallel to slope

Perpendicular to slope

Normal reaction + Tension + Weight = Resultant

(iii) Once you know the resultant force you can work out the acceleration of the sledge using Newton's second law.

$$F = ma$$
$$21.959\ldots = 5a$$
$$a = \frac{21.959\ldots}{5} = 4.392\ldots$$

The acceleration is $4.4\,\mathrm{m\,s^{-2}}$ (correct to 1 d.p.)

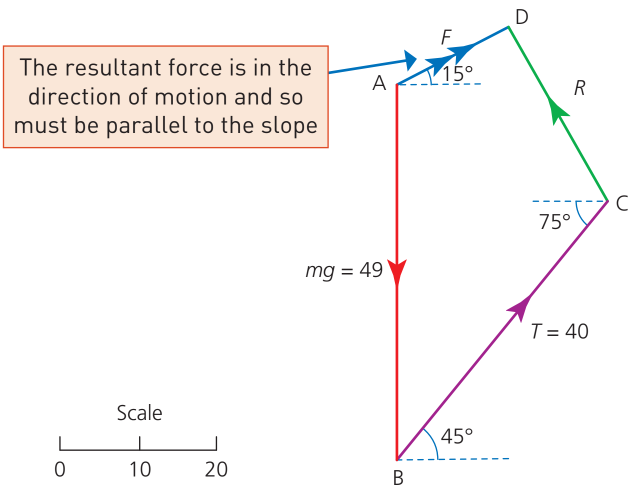


Figure 3.68

An alternative way of approaching the previous example is to draw a scale diagram with the three forces represented by three of the sides of a quadrilateral taken in order (with the arrows following each other, $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CD}$) as shown in Figure 3.68. The resultant is represented by the fourth side AD.

From the diagram you can estimate the normal reaction to be about 30 N and the resultant 20 N.

Discussion point

In what order would you draw the lines in the diagram?

Discussion point

What can you say about the acceleration of the sledge in the cases when

(i) the length AD in Figure 3.68 is not zero?

(ii) the length AD is zero so that the starting point on the quadrilateral is the same as the finishing point?

(iii) BC is so short that the point D is to the left of A as shown in Figure 3.69?

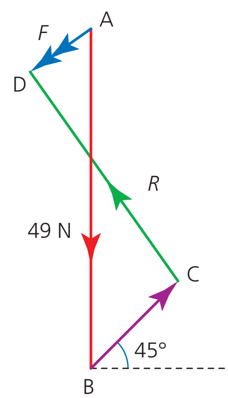


Figure 3.69

Example 3.15

Two forces **P** and **Q** act at a point O on a particle of mass 2 kg. Force **P** has magnitude 50 N and acts along a bearing of 030°. Force **Q** has magnitude of 30 N and acts along a bearing of 315°.

(i) Find the magnitude and bearing of the resultant force **P** + **Q.**

(ii) Find the acceleration of the particle.

Solution

(i) Forces **P** and **Q** are illustrated below.

> **Note**
>
> Notice that **P** and **Q** are written as vectors.

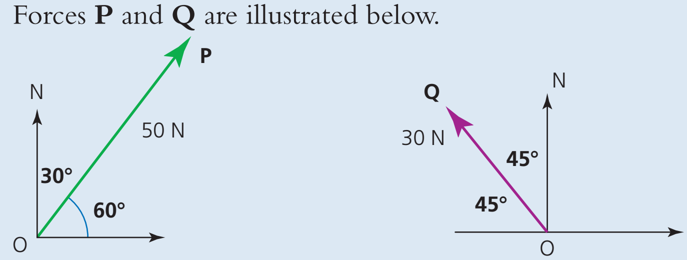


Figure 3.70

$$\mathbf{P} = \begin{pmatrix} 50\cos 60° \\ 50\sin 60° \end{pmatrix}$$
$$= \begin{pmatrix} 25 \\ 43.30\ldots \end{pmatrix}$$
$$\mathbf{Q} = \begin{pmatrix} -30\cos 45° \\ 30\sin 45° \end{pmatrix}$$
$$= \begin{pmatrix} -21.21\ldots \\ 21.21\ldots \end{pmatrix}$$
$$\mathbf{P}+\mathbf{Q} = \begin{pmatrix} 25 \\ 43.30\ldots \end{pmatrix} + \begin{pmatrix} -21.21\ldots \\ 21.21\ldots \end{pmatrix}$$
$$= \begin{pmatrix} 3.78\ldots \\ 64.51\ldots \end{pmatrix}$$

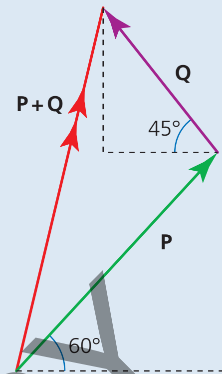


Figure 3.71

The resultant is shown in Figure 3.72.

Magnitude $|\mathbf{P}+\mathbf{Q}| = \sqrt{3.78\ldots^2 + 64.51\ldots^2}$
$= 64.62$

Direction $\tan\theta = \dfrac{64.51\ldots}{3.78\ldots}$
$\theta = 86.64\ldots°$

The bearing is $90° - 86.64\ldots° = 3.36\ldots°$

The force $\mathbf{P} + \mathbf{Q}$ has magnitude 65 N and bearing 003°.

(ii) The acceleration of the particle is given by

$$\mathbf{a} = \tfrac{1}{2}(\mathbf{P}+\mathbf{Q}) = \frac{1}{2}\begin{pmatrix} 3.78\ldots \\ 64.51\ldots \end{pmatrix} = \begin{pmatrix} 1.89 \\ 32.3 \end{pmatrix}$$

$m = 2\,\text{kg}$

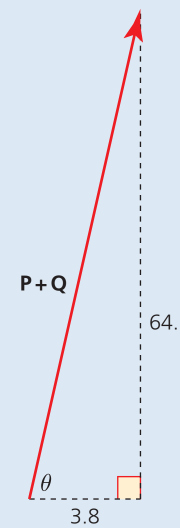


Figure 3.72

The magnitude of the acceleration is $\sqrt{1.89^2 + 32.3^2} = 32.4$.

The bearing is $\arctan\left(\dfrac{1.89}{32.3}\right) = 3.35°$.

P and **Q** give the particle an acceleration of 32.3 m s^{-2} on a bearing of 003°.

Sometimes, as in the next example, it is just as easy to work with the trigonometry of the diagram as with the components of the forces.

Example 3.16

The angle between the lines of action of two forces **X** and **Y** is θ. Find the magnitude and direction of the resultant.

Solution

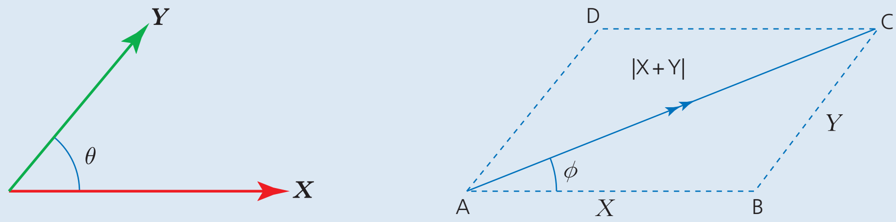


Figure 3.73

Use the cosine rule in triangle ABC. The magnitude of the resultant is F.

$$F = |\mathbf{X} + \mathbf{Y}| = \sqrt{\mathrm{AB}^2 + \mathrm{BC}^2 - 2\mathrm{AB} \times \mathrm{BC} \times \cos(\widehat{\mathrm{ABC}})}$$

$$= \sqrt{X^2 + Y^2 - 2XY\cos(180° - \theta)}$$

$$= \sqrt{X^2 + Y^2 + 2XY\cos\theta}$$

Use the sine rule in triangle ABC. The resultant makes an angle ϕ with the **X** force.

$$\frac{\sin\widehat{\mathrm{CAB}}}{\mathrm{BC}} = \frac{\sin\widehat{\mathrm{ABC}}}{\mathrm{AC}}$$

$$\frac{\sin\phi}{Y} = \frac{\sin(180° - \theta)}{F}$$

$$\sin\phi = \frac{Y}{F}\sin\theta$$

$$\phi = \arcsin\left(\frac{Y}{F}\sin\theta\right)$$

The resultant of the two forces **X** and **Y** inclined at θ has magnitude $F = \sqrt{X^2 + Y^2 + 2XY\cos\theta}$ and makes an angle $\arcsin\left(\frac{Y}{F}\sin\theta\right)$ with the **X** force.

Exercise 3.4

For questions 1–6, carry out the following steps. All forces are in newtons.

(i) Draw a scale diagram to show the forces and their resultant.

(ii) State whether you think the forces are in equilibrium and, if not, estimate the magnitude and direction of the resultant.

(iii) Write the forces in component form, using the directions indicated and so obtain the components of the resultant. Hence find the magnitude and direction of the resultant.

(iv) Compare your answers to parts (ii) and (iii).

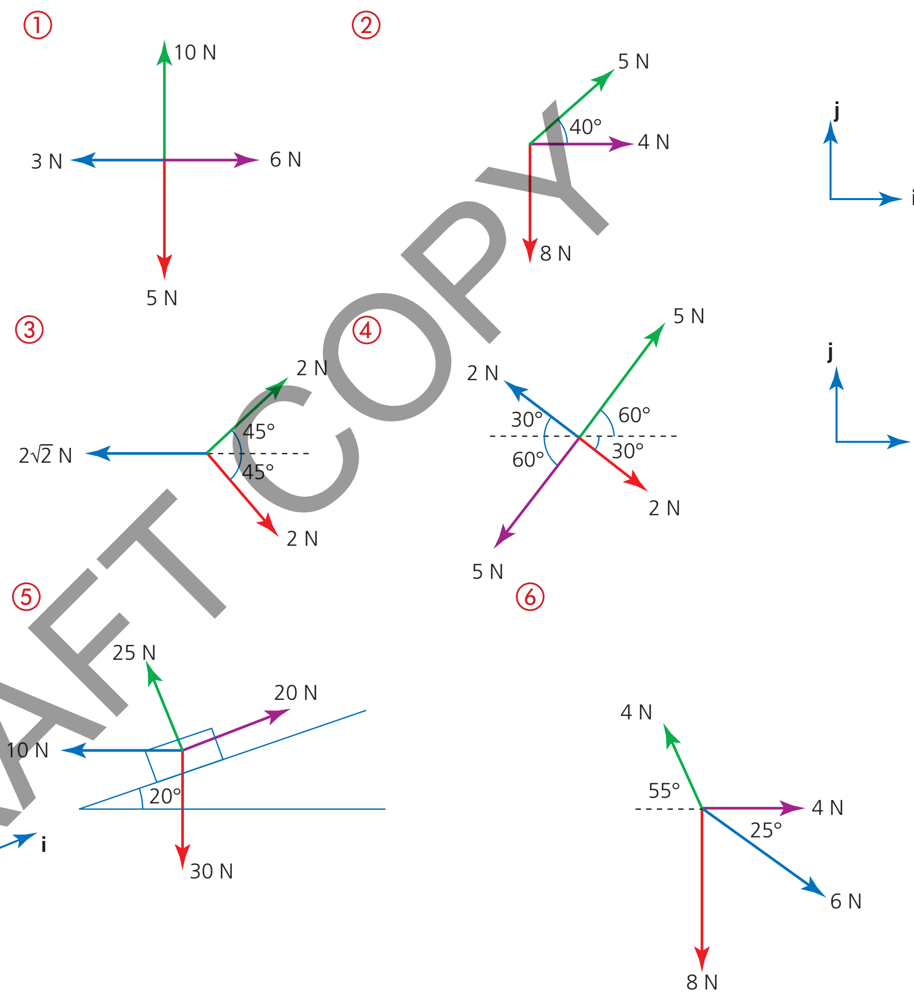


Figure 3.74

⑦ Four horizontal wires are attached to a telephone post and exert the following tensions on it: 25 N in the north direction, 30 N in the east direction, 45 N in the north-west direction and 50 N in the south-west direction. Calculate the resultant tension on the post and find its direction.

8 Forces of magnitude 7 N, 10 N and 15 N act on a particle of mass 1.5 kg in the directions shown in Figure 3.75.

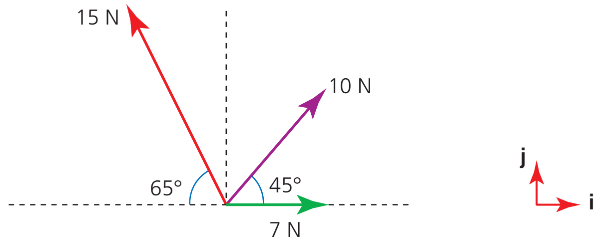


Figure 3.75

(i) Find the components of the resultant of the three forces in the **i** and **j** directions.

(ii) Find the magnitude and direction of the resultant.

(iii) Find the acceleration of the particle.

9 (i) Find the resultant of the set of 6 forces whose magnitudes and directions are shown in Figure 3.76.

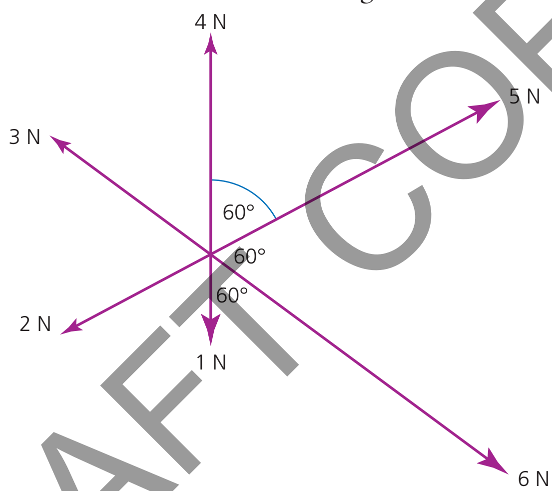


Figure 3.76

The forces are acting on a particle P of mass 5 kg which is initially at rest at O.

(ii) How fast is P moving after 3 s and how far from O is it now?

10 The resultant of two forces **P** and **Q** acting on a particle has magnitude $P = |\mathbf{P}|$. The resultant of the two forces $3\mathbf{P}$ and $2\mathbf{Q}$ acting in the same directions as before has magnitude $2P$. Find the magnitude of **Q** and the angle between **P** and **Q**.

KEY POINTS

1 There are many different types of force, including gravity, tension, thrust, driving force and friction.

2 Newton's laws of motion
- Every object continues in a state of rest or uniform motion in a straight line unless it is acted on by an external force.
- Resultant force = mass × acceleration or $\mathbf{F} = m\mathbf{a}$
- When one object exerts a force on another there is always a reaction force which is equal and opposite in direction, to the acting force.

3 Vector quantities (like force) can be added to find their resultant, resolved into components, and their magnitude and direction can be calculated.

4 Relationships between the variables describing motion

Position	Velocity	Acceleration
→	differentiate	→
$\mathbf{r} = x\mathbf{i} + y\mathbf{j}$	$\mathbf{v} = \frac{d\mathbf{r}}{dt} = v_x\mathbf{i} + v_y\mathbf{j}$	$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = a_x\mathbf{i} + a_y\mathbf{j}$
$\mathbf{r} = \begin{pmatrix} x \\ y \end{pmatrix}$	$\mathbf{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$	$\mathbf{a} = \begin{pmatrix} \frac{dv_x}{dt} \\ \frac{dv_y}{dt} \end{pmatrix} = \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix}$
Distance from O $\sqrt{\dot{x}^2 + \dot{y}^2}$	Speed $\sqrt{x^2 + y^2}$	Magnitude of acceleration $\sqrt{\ddot{x}^2 + \ddot{y}^2}$

Acceleration	Velocity	Position
→	integrate	→
$\mathbf{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$	$\mathbf{v} = \int \mathbf{a}\, dt$	$\mathbf{r} = \int \mathbf{v}\, dt$
	$\mathbf{v} = \int \begin{pmatrix} a_x \\ a_y \end{pmatrix} dt$	$\mathbf{r} = \int \begin{pmatrix} v_x \\ v_y \end{pmatrix} dt$

$$2\mathbf{a} \bullet (\mathbf{r} - \mathrm{r}) = v^2 - u^2$$

6 Force is a vector. It may be represented in either magnitude-direction form or in component form

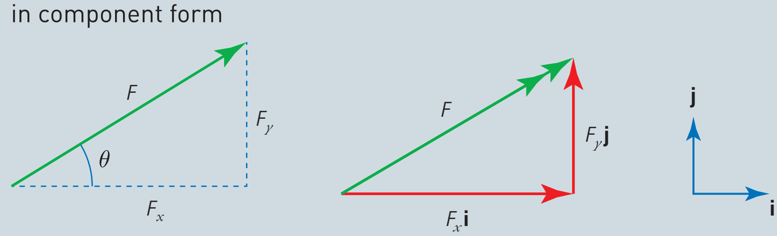


Figure 3.77

Note

The magnitude of any vector, say $\mathbf{p}$, is denoted by p or $|\mathbf{p}|$. This is given by $p = \sqrt{p_x^2 + p_y^2 + p_z^2}$ and so $p^2 = p_x^2 + p_y^2 + p_z^2 = \mathbf{p}.\mathbf{p}$.

In this case $\mathbf{v}.\mathbf{v} = v^2$ and $\mathbf{u}.\mathbf{u} = u^2$.

Given a vector $a = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$ at an angle of θ with the x axis, then

- If a_x is positive then $\theta = \arctan \begin{pmatrix} a_y \\ a_x \end{pmatrix}$
- If a_x is negative and a_y is positive then $\theta = \arctan \begin{pmatrix} a_y \\ a_x \end{pmatrix} + 180$ (or in radians this is $\begin{pmatrix} a_y \\ a_x \end{pmatrix} + \pi$)
- If a_x is negative and a_y is negative then $\theta = \arctan \begin{pmatrix} a_y \\ a_x \end{pmatrix} - 180$ (or in radians this is $\begin{pmatrix} a_y \\ a_x \end{pmatrix} - \pi$).

Magnitude of $\mathbf{F} = |\mathbf{F}| = \sqrt{F_x^2 + F_y^2}$ $\quad \mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} = \begin{pmatrix} F_x \\ F_y \end{pmatrix}$

Direction of $\mathbf{F} = \theta = \arctan\left(\frac{F_y}{F_x}\right)$

7 Resolving forces

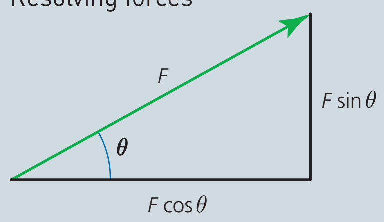


Figure 3.78

$$\mathbf{F} = F\cos\theta\mathbf{i} + F\sin\theta\mathbf{j} = \begin{pmatrix} F\cos\theta \\ F\sin\theta \end{pmatrix}$$

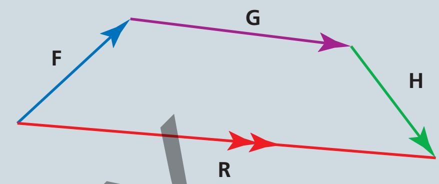


Figure 3.79

8 Resultant forces

$$\mathbf{R} = \mathbf{F} + \mathbf{G} + \mathbf{H} = \begin{pmatrix} F_x \\ F_y \end{pmatrix} + \begin{pmatrix} G_x \\ G_y \end{pmatrix} + \begin{pmatrix} H_x \\ H_y \end{pmatrix}$$

$$\mathbf{R} = \begin{pmatrix} X \\ Y \end{pmatrix};\ X = F_x + G_x + H_x,\ Y = F_y + G_y + H_y$$

Magnitude of $\mathbf{R} = \sqrt{X^2 + Y^2}$ $\quad$ Direction of $\mathbf{R} = \theta = \arctan\left(\frac{Y}{X}\right)$

9 Equilibrium

When the resultant is zero, the forces are in equilibrium.

10 Triangle of forces

If an object is in equilibrium under three non-parallel forces, their lines of action are concurrent and they can be represented by a triangle.

LEARNING OUTCOMES

When you have finished this chapter, you should be able to:

- draw a diagram showing the forces acting on a body
- apply Newton's laws of motion to problems in one or more dimensions
- resolve a force into components having selected suitable directions for resolution
- find the resultant of several concurrent forces
- realise that a particle is in equilibrium under a set of concurrent forces if and only if the resultant is zero
- know that a closed polygon may be drawn to represent the forces acting on a particle in equilibrium
- formulate equations for equilibrium by resolving forces in suitable directions
- formulate the equation of motion of a particle which is being acted on by several forces
- know that contact between two surfaces is lost when the normal reaction force becomes zero
- work with vectors in two dimensions.

4 A model for friction

A gem cannot be polished without friction, nor a man perfected without trials

Lucius Annaeus Seneca

Figure 4.1 When an accident involving a collision happens, skid marks are sometimes left on the roadway by a vehicle that has locked its brakes. By measuring the skid marks and applying mechanics, it is possible to estimate the speed of the vehicle, prior to collision.

Discussion point

In the situation illustrated in Figure 4.1, the red car left a 10 metre skid mark on the road. The driver of the car claimed that this showed he was driving within the speed limit of 30 mph.

It is the duty of a court to decide whether the driver of the red car was innocent or guilty. Is it possible to deduce his speed from the skid mark? Draw a sketch map and make a list of the important factors that you would need to consider when modelling this situation.

1 A model for friction

Clearly the key information about the accident involving the red car is provided by the skid marks. To interpret it, you need a model for how friction works; in this case between the car's tyres and the road.

As a result of experimental work, Coulomb formulated a model for friction between two surfaces. The following laws are usually attributed to him.

1. Friction always opposes relative motion between two surfaces in contact.
2. Friction is independent of the relative speed of the surfaces.
3. The magnitude of the frictional force has a maximum which depends on the normal reaction between the surfaces and on the roughness of the surfaces in contact.
4. If there is no sliding between the surfaces

   $F \leq \mu R$

   where F is the force due to friction and R is the normal reaction. μ is called the *coefficient of friction*.
5. When sliding is just about to occur, friction is said to be *limiting* and $F = \mu R$.
6. When sliding occurs $F = \mu R$.

According to Coulomb's model, μ is a constant for any pair of surfaces. Typical values and ranges of values for the coefficient of friction μ are given in the table.

Surfaces in contact	μ
Wood sliding on wood	0.2–0.6
Metal sliding on metal	0.15–0.3
Normal tyres on dry road	0.8
Racing tyres on dry road	1.0
Sandpaper on sandpaper	2.0
Skis on snow	0.02

How fast was the driver of the red car going?

You can proceed with the problem. As an initial model, the driver of the red car made the following assumptions:

1. that the road was level
2. that his car was travelling at $2\,\mathrm{m\,s^{-1}}$ when it hit the orange car (this was consistent with the damage to the cars)
3. that the car and driver could be treated as a particle, subject to Coulomb's laws of friction with $\mu = 0.8$ (i.e. dry road conditions).

Discussion point

How good is this model and would you be confident in offering the answer as evidence in your defence in court? Look carefully at the three assumptions. What effect do they have on the estimate of the initial speed?

Taking the direction of travel as positive, let the car and driver have acceleration $a\,\mathrm{m\,s^{-2}}$ and mass $m\,\mathrm{kg}$. You have probably realised that the acceleration will be negative. The forces (in N) and acceleration are shown in Figure 4.2.

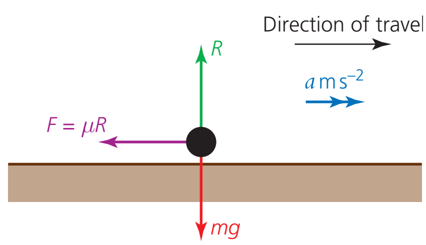


Figure 4.2

Apply Newton's second law:

perpendicular to the road $R - mg = 0$ ①

parallel to the road $-\mu R = ma$ ②

There is no vertical acceleration

There is a constant force $-\mu R = ma$ from friction

Solving for a gives

$$a = -\frac{\mu R}{m} = -\frac{\mu mg}{m} = -\mu g$$

From ① $R = mg$

Taking $\mu = 0.8$ and $g = 9.8\,\mathrm{m\,s^{-2}}$ gives $a = -7.84\,\mathrm{m\,s^{-2}}$.

The constant acceleration formula

$$v^2 = u^2 + 2as$$

can be used to calculate the initial speed of the red car. Substituting $s = 10$, $v = 2$ and $a = -7.84$ gives

$$u = \sqrt{2^2 + 2 \times 7.84 \times 10} = 12.68\,\mathrm{m\,s^{-1}}$$

Convert this figure to miles per hour.

$$\text{Speed} = \frac{12.68 \times 3600}{1600}$$

$$= 28.5\,\text{mph}$$

1 hour is 60 × 60 seconds

1 mile is approx. 1600 m

So the model suggests that the red car was travelling at a speed of just under 30 mph before skidding began.

Discussion point

In this example the brakes were locked. What happens when you slow down in normal driving? Where does friction act?

2 Modelling with friction

While there is always some frictional force between two sliding surfaces its magnitude is often so small as to be negligible. In such cases the surfaces are described as *smooth*.

In situations where frictional forces cannot be ignored, the surface(s) are described as *rough*. Coulomb's law is the standard model for dealing with such cases.

Frictional forces are essential in many ways. For example, a ladder leaning against a wall would always slide if there were no friction between the foot of the ladder and the ground. The absence of friction in icy conditions causes difficulties for road users: pedestrians slip over, cars and motorcycles skid.

Remember that friction always opposes sliding motion.

Discussion point

In what direction is the frictional force between the back wheel of a cycle and the road?

Example 4.1

A stone slides in a straight line across a frozen pond. It starts to move with a speed of $8\,\mathrm{m\,s^{-1}}$ and slides for 40 m before coming to rest. Calculate the coefficient of friction between the stone and the pond.

Solution

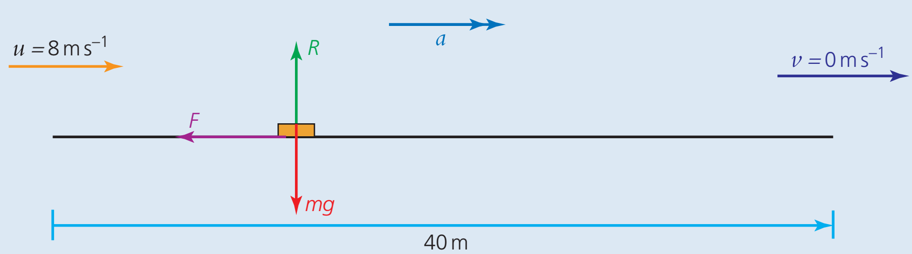


Figure 4.3

The only force acting on the stone in the direction of motion is the frictional force F, which is uniform, thus giving rise to a constant acceleration a. In order to find a, use the constant acceleration formula

$$v^2 = u^2 + 2as$$

$$0 = 8^2 + 2 \times 40 \times a$$

Since the stone is in motion: $F = \mu R$

$$a = -\frac{64}{80} = -0.8$$

Use Newton's second law to give

$$-F = -0.8m$$

$$-\mu mg = -0.8m$$

Vertical equilibrium means $R = mg$

$$\mu = \frac{0.8}{9.8} = 0.0816\ldots$$

The coefficient of friction between the stone and the pond is 0.082.

Example 4.2

A box of mass 2 kg rests on rough horizontal ground. The coefficient of friction between the box and the ground is 0.4. A light inextensible string is attached to the box in order to pull it along. If the tension in the string is T N, find the value that T must exceed if the box is to accelerate when the string is

(i) horizontal

(ii) 30° above the horizontal

(iii) 30° below the horizontal.

Solution

(i) The forces acting on the box are shown in Figure 4.4.

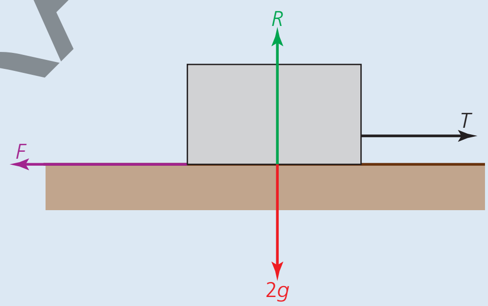


Figure 4.4

Horizontal forces: $T > F$

This is the condition for the box to accelerate

Vertical forces: $R = 2g$

$$R = 2 \times 9.8 = 19.6$$

The law of friction states that for a moving object

$F = \mu R$

So in this case

$F = 0.4 \times 19.6$

$F = 7.84\,\text{N}$

T must exceed 7.84 N for the box to accelerate.

(ii) The forces acting on the box are shown in Figure 4.5.

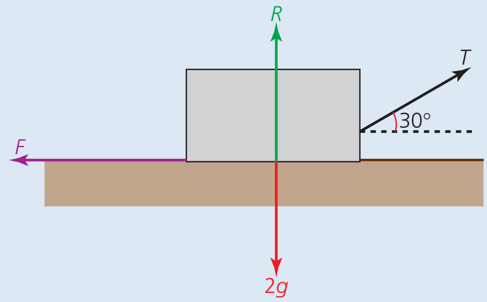


Figure 4.5

Resolving horizontally: $T\cos 30° > F$ ①

This is the condition for the box to accelerate

vertically: $R + T\sin 30° = 2g$

So that $R = 2g - T\sin 30°$

This equation shows that in this situation the magnitude of R is less than $2g$

When motion occurs $F = \mu R = 0.4(2g - T\sin 30°)$

Substituting in ① $T\cos 30° > 0.4(2g - T\sin 30°)$

Rearranging $T(\cos 30° + 0.4\sin 30°) > 0.8g$

$$T > \frac{0.8 \times 9.8}{(\cos 30° + 0.4\sin 30°)} = \frac{7.84}{1.066} = 7.35\ \ldots$$

T must exceed 7.35 N for the box to accelerate.

(iii) The forces acting on the box are shown in Figure 4.6.

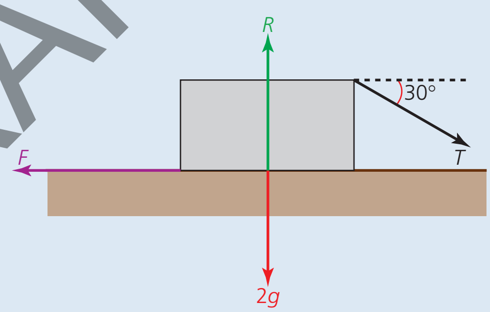


Figure 4.6

Resolving horizontally: $T\cos 30° > F$

vertically: $R = T\sin 30° + 2g$

When motion occurs $F = 0.4R = 0.4\,(T\sin 30° + 2g)$

So that $T\cos 30° > 0.4\,(T\sin 30° + 2g)$

$$T > \frac{7.84}{(\cos 30° - 0.4\sin 30°)} = \frac{7.84}{0.666}$$

$$T > 11.77\ \ldots$$

T must exceed 11.8 N for the box to accelerate.

Note

It is clear from the example that the force required to move the box is largest when the force from the string is pointing downwards. The component of the force in the downward direction increases the normal reaction, which in turn increases the frictional force.

Example 4.3

Figure 4.7 shows a block of mass 5 kg resting on a rough table and connected by a light inextensible string passing over a smooth pulley to a block of mass 4 kg. The coefficient of friction between the 5 kg block and the table is 0.4.

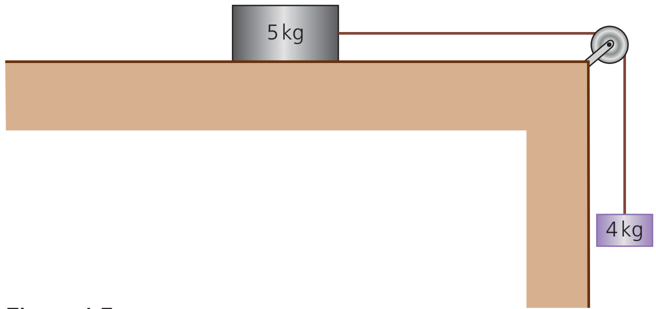


Figure 4.7

(i) Draw diagrams showing the forces acting on each block and the direction of the system's acceleration.

(ii) Show that acceleration does take place.

(iii) Find the acceleration of the system and the tension in the string.

Solution

(i)

The 5 kg block has no vertical acceleration. $R = 5g$

The directions of the acceleration of the blocks are clearly as shown here

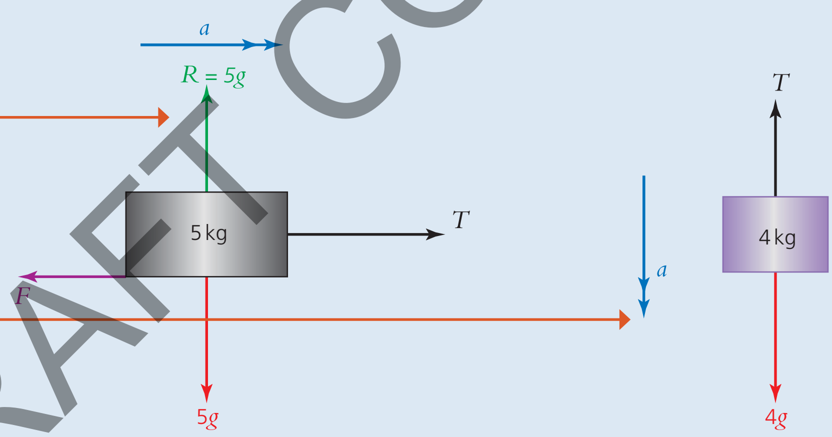


Figure 4.8

(ii) The equations of motion of the blocks are

The 4 kg block $\quad 4g - T = 4a \quad$ ①

The 5 kg block $\quad T - F = 5a \quad$ ②

Adding ① and ② $\quad 4g - F = 9a \quad$ ③

You need to show that $a > 0$

The maximum possible value of F is $\mu R = 0.4 \times 5g = 2g$

So in the left hand side of ③, $4g - F > 0$

Therefore $a > 0$ and sliding occurs.

(iii) When sliding occurs, you can replace F by $\mu R = 2g$

Then ③ gives $2g = 9a$

$$a = \frac{2}{9}g = 2.17$$

Substituting in ① gives $T = 4(g - a) = 4 \times \frac{7}{9}g = 30.5$ (3 s.f.)

The final answer is rounded to 3 s.f. and so is consistent with a value of 9.80 for g.

The acceleration of the system is $2.2\,\mathrm{ms^{-2}}$ and the tension in the string is 30.5 N.

Example 4.4

A block of mass m is placed on a slope inclined at an angle θ to the horizontal. The coefficient of friction between the block and the slope is 0.4.

(i) Find the values of θ for which the block would be at rest.

The angle of the slope is actually 30°.

(ii) Find the time taken for the block to slide a distance of 1.5 m down the slope, assuming it starts at rest.

Solution

(i) The forces acting on the block are shown in Figure 4.9.

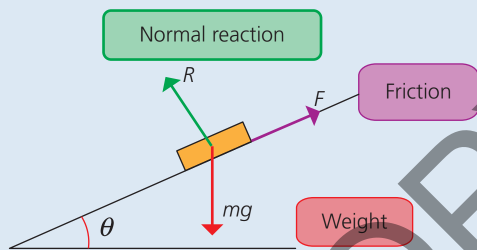


Figure 4.9

The weight mg can be resolved into components $mg\cos\theta$ perpendicular to the slope and $mg\sin\theta$ parallel to the slope.

Note

In a situation like this where an object is on a slope, it is almost always easier to work with directions perpendicular and parallel to the slope rather than vertical and horizontal.

Note

You can think of the weight mg as the resultant of two resolved components.

Since the block is in equilibrium

Parallel to the slope: $F = mg\sin\theta$ ①

Perpendicular to the slope: $R = mg\cos\theta$ ②

Since the block is at rest: $F \leq \mu R$

Substituting for F and R from ① and ②

$$mg\sin\theta \leq \mu mg\cos\theta$$

$$\Rightarrow \sin\theta \leq \mu\cos\theta$$

$$\Rightarrow \tan\theta \leq \mu$$

$$\theta \leq \arctan\mu$$

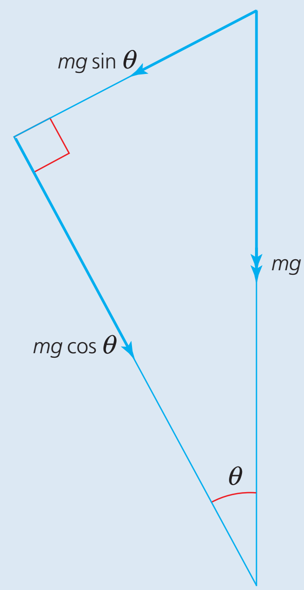


Figure 4.10

In this case $\mu = 0.4$, so $\tan\theta \leq 0.4$ and $\theta \leq 21.8°$.

The block is at rest for values of θ less than 21.8°.

(ii)

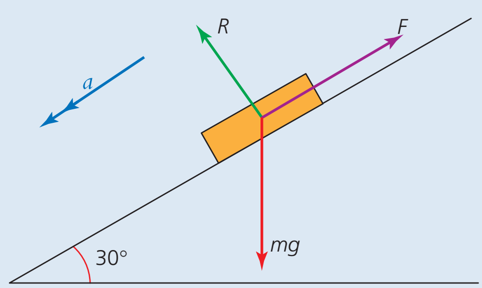


Figure 4.11

The block is now sliding down the plane. It has an acceleration, a.

The equation of motion for the block is

$$ma = mg\sin 30° - F \quad ③$$

Since the block is moving

$$F = \mu R = 0.4mg\cos 30° \quad ④$$

Substituting ④ into ③

$$ma = mg\sin 30° - 0.4mg\cos 30°$$

Dividing through by m

$$a = g(\sin 30° - 0.4\cos 30°)$$
$$= 9.8 \times (0.5 - 0.4 \times 0.866\ldots)$$
$$= 1.505\ldots$$

Using the constant acceleration formula $s = ut + \frac{1}{2}at^2$ with $u = 0$, $s = 1.5$ and $a = 1.505$..

$$1.5 = 0.5 \times 1.505\ldots \times t^2$$

$$\Rightarrow t = \sqrt{\frac{1.5}{0.5 \times 1.505\ldots}} = 1.41\ldots$$

The block takes 1.4 s to slide down the slope.

Historical note

Charles Augustin de Coulomb was born in Angoulême in France in 1736 and is best remembered for his work on electricity rather than that on friction. Coulomb's law concerns the forces on charged particles and was determined using a torsion balance. The unit for electric charge is named after him. Coulomb worked in many fields, including the elasticity of metal, silk fibres and the design of windmills. He died in Paris in 1806.

Note

1 The result is independent of the mass of the block. This is often found when simple models are applied to mechanics problems. For example, two objects of different mass fall to the ground with the same acceleration. However when such models are refined, for example, to take account of air resistance, mass is often found to have some effect on the result.

2 The angle for which the block is about to slide down the slope is called the *angle of friction*. The angle of friction is often denoted by λ (lambda) and is defined by $\tan\lambda = \mu$.
When the angle of the slope is equal to the angle of the friction, it is just possible for the block to stay on the slope without sliding. If the slope is slightly steeper, the block will start to slide.

Exercise 4.1

① A block of mass 25 kg is resting on a horizontal surface. It is being pulled by a horizontal force T N, and is on the point of sliding. Draw a diagram showing the forces acting and find the coefficient of friction when

(i) $T = 20$

(ii) $T = 5$

② A box of mass 25 kg is resting on rough horizontal ground. The box can just be moved by a horizontal force of 60 N. Find the coefficient of friction between the box and the floor.

③ A stone is sliding across a frozen pond. It travels a distance of 12 m before coming to rest from an initial speed of 4 m s^{-1}.

Find the coefficient of friction between the stone and the pond.

④ A parcel drops out of a van travelling at 20 m s^{-1}. The parcel slides a distance of 30 m before coming to rest. Calculate the coefficient of friction between the parcel and the road.

⑤ In each of the following situations find, in any order, the acceleration of the system, the tension(s) in the string(s) and the magnitude of the frictional force.

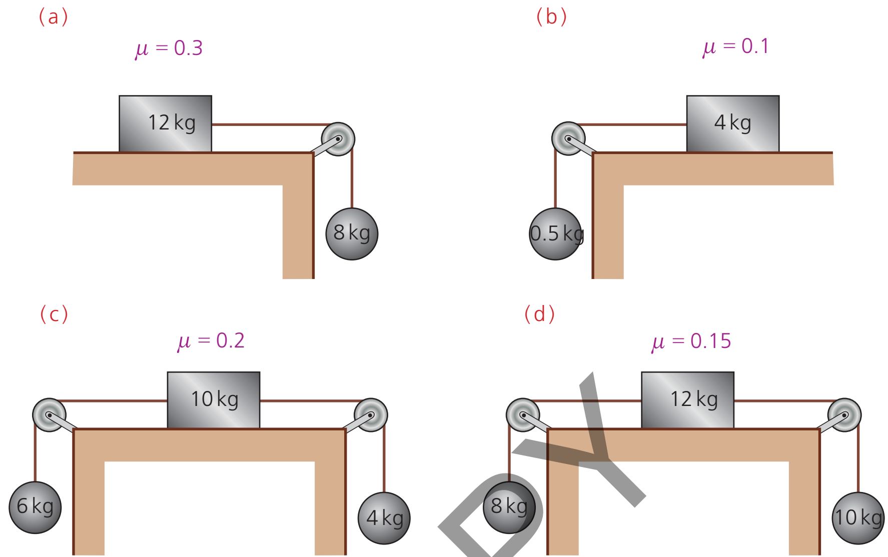


Figure 4.12

⑥ A block of mass 5 kg is about to move up a rough plane inclined at 30° to the horizontal, under the application of a force of 50 N parallel to the slope, as shown in the diagram.

Find the value of the coefficient of friction between the block and the plane.

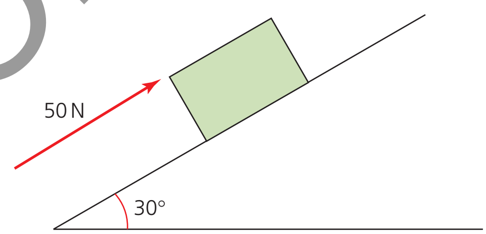


Figure 4.13

⑦ An ice hockey player is sliding a puck of mass 50 g across the ice rink. The initial speed of the puck is $8\,\text{m s}^{-1}$ and it takes it 40 m to come to rest.

(i) Find the deceleration of the puck.

(ii) Find the frictional force acting on the puck.

(iii) Find the coefficient of friction between the puck and the ice rink.

(iv) How far will a 60 g puck travel if it, too, is given an initial speed of $8\,\text{m s}^{-1}$?

⑧ A car of mass 1200 kg is travelling at $30\,\text{m s}^{-1}$ when it is forced to perform an emergency stop. Its wheels lock as soon as the brakes are applied so that they slide along the road without rotating. For the first 40 m the coefficient of friction between the wheels and the road is 0.75 but then the road surface changes and the coefficient of friction becomes 0.8.

(i) Find the deceleration of the car immediately after the brakes are applied.

(ii) Find the speed of the car when it comes to the change of road surface.

(iii) Find the total distance the car travels before it comes to rest.

⑨ A girl, whose mass is 25 kg, is sitting on a sledge of mass 10 kg which is being pulled at constant speed along horizontal ground by her brother. The coefficient of friction between the sledge and the snow-covered ground is 0.1. Find the tension in the rope from the boy's hand to the sledge when:

(i) the rope is horizontal

(ii) the rope makes an angle of 20° with the horizontal.

⑩ A particle of mass 5 kg is projected upwards along a plane that is inclined at an angle of 25° to the horizontal, with a speed of $10\,\text{m s}^{-1}$. The particle comes to rest after 10 m.

(i) Find the deceleration of the particle.

(ii) Find the frictional force F and the normal reaction R, and hence deduce the coefficient of friction between the particle and the plane.

The particle then starts to move down the plane with acceleration $a\ \text{m s}^{-2}$.

(iii) Find a and the speed of the particle as it passes its starting point.

⑪ A 10 kg block lies on a rough horizontal table. The coefficient of friction between the block and the table is 0.15. The block is attached, by a light inextensible string, which passes over a smooth pulley to a mass of 2 kg hanging freely. The 10 kg block is 1.2 m from the pulley and the 2 kg mass is 1 m from the floor. The system is released from rest. Find:

(i) the acceleration of the system

(ii) the time taken for the 2 kg mass to reach the floor

(iii) the velocity with which the 10 kg mass hits the pulley.

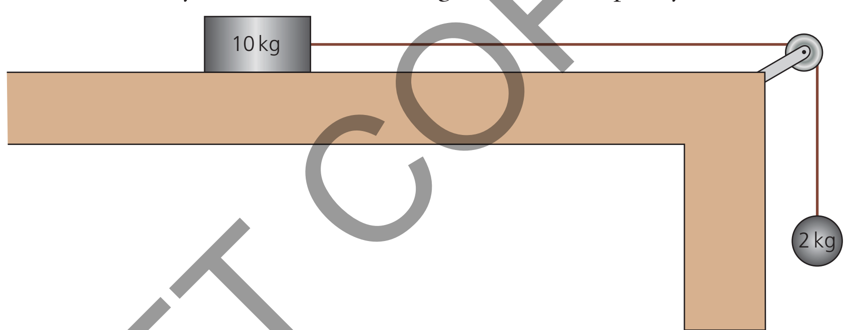


Figure 4.14

⑫ A box of weight 200 N is pulled at a steady speed across a rough horizontal surface by a rope which makes an angle α with the horizontal. The coefficient of friction between the box and the surface is 0.5. Assume that the box slides on its underside and does not tip up.

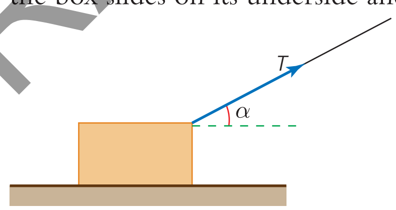


Figure 4.15

(i) Find an expression for the value of T for any angle α.

(ii) For what value of α is T a minimum and what is the value of that minimum?

⑬ A block of mass 10 kg is lying on a rough plane inclined at 30° to the horizontal. A horizontal force P is applied to the block as shown in Figure 4.16. The coefficient of friction between the block and the plane is 0.5.

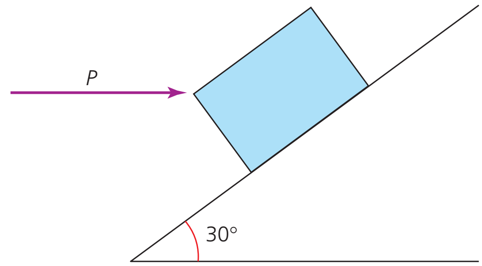


Figure 4.16

(i) Find the least force P necessary to start the block sliding up the plane.

(ii) Find the least force P necessary to stop the block from sliding down the plane.

KEY POINTS

1 The total contact force between two surfaces may be expressed in terms of a frictional force and a normal reaction.

2 The frictional force, F, between two surfaces is given by:
$F < \mu R$ when there is no sliding except in limiting equilibrium
$F = \mu R$ in limiting equilibrium
$F = \mu R$ when sliding occurs
where R is the normal reaction of one surface on the other and μ is the coefficient of friction between the surfaces.

3 The frictional force always acts in the direction to oppose sliding.

4 The magnitude of the normal reaction is affected by any force which has a component perpendicular to the direction of sliding.

LEARNING OUTCOMES

When you have completed this chapter, you should:

- understand that the total contact force between surfaces may be expressed in terms of a frictional force and a normal reaction
- be able to draw a force diagram to represent a situation involving friction
- understand that a frictional force may be modelled by $F \leq \mu R$
- know that a frictional force acts in the direction to oppose sliding
- be able to model friction using $F = \mu R$ when sliding occurs
- know how to apply Newton's laws of motion to situations involving friction
- be able to derive and use the result that a body on a rough slope inclined at angle α to the horizontal is on the point of slipping if $\mu = \tan\alpha$.

PRACTICE QUESTIONS: SET 1

① A train starts from rest and accelerates uniformly for 4 minutes, by which time it has gained a speed of $36\,\text{km}\,\text{h}^{-1}$. It runs at this speed for 5 minutes and then decelerates uniformly, coming to rest in 2 minutes.

(i) Draw the speed-time graph using consistent units on the two axes. [4 marks]

(ii) Find the total distance travelled. [3 marks]

② A particle is moving in a straight line. The position s of the particle at time t is given by

$$s = 18 - 24t + 9t^2 - t^3, \quad 0 \le t \le 5$$

(i) Find the velocity v at time t and the values of t for which $v = 0$. [4 marks]

(ii) Find the position of the particle at those times. [2 marks]

(iii) Find the total distance travelled by the particle in the interval $0 \le t \le 5$. [5 marks]

③ A skier of mass 60 kg is skiing down a slope inclined at 20° to the horizontal.

(i) Draw a diagram showing the forces acting on the skier. [3 marks]

(ii) Resolve these forces into components parallel and perpendicular to the slope. [2 marks]

(iii) The skier is travelling at constant speed. Find the normal reaction of the slope on the skier and the resistive force on her. [3 marks]

The skier later returns to the top of the slope by being pulled up it at a constant speed of $8\,\text{m}\,\text{s}^{-1}$ by a rope parallel to the slope.

(iv) Assuming the resistance on the skier is the same as before, calculate the tension in the rope. [2 marks]

(v) The rope suddenly breaks. How far does the skier travel before coming to rest? [4 marks]

④ Two forces **P** and **Q** act in the directions of the unit vectors **i** and **j**. The resultant has magnitude 20 N and makes an angle of 60° with the direction of **i**.

(i) Find the magnitude of **P** and **Q**. [3 marks]

(ii) Find the magnitude and direction of a force of $-3\mathbf{P} + \sqrt{3}\mathbf{Q}$. [3 marks]

⑤ Figure 1 below shows the path of a particle P moving in the Cartesian plane with origin O, drawn with graph-drawing software.

The position of P at time t s is $\mathbf{r} = \left(2t^3 - 3t^2 + 2\right)\mathbf{i} + (t-1)^2\mathbf{j}$, where **i** and **j** are unit vectors in the directions Ox and Oy and $-1 \le t \le 2$.

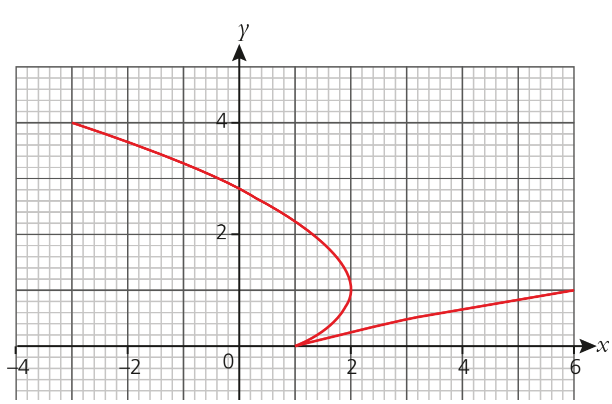


Figure 1

(i) Determine the time(s), if any, when P is instantaneously at rest. [3 marks]

(ii) Sketch a copy of the path of P and indicate the points where $t = -1, 0, 1$ and 2 and the direction of travel.

Describe briefly the motion of P about the time when $t = 0$.

Describe briefly the motion of P about the time when $t = 1$. [4 marks]

⑥

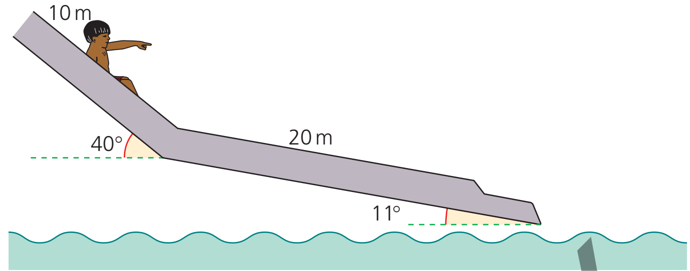


Figure 2

A chute at a water sports centre has been designed so that swimmers first slide down a steep part which is 10 m long and at an angle of 40° to the horizontal. They then come to a 20 m section with a gentler slope, 11° to the horizontal, where they travel at constant speed.

(i) Find the coefficient of friction between a swimmer and the chute. [4 marks]

(ii) Find the acceleration of a swimmer on the steeper part. [4 marks]

(iii) Find the speed of a swimmer at the end of the chute. (You may assume that no speed is lost at the point where the slope changes). [3 marks]

An alternative design of chute is made out of the same material. It has the same starting and finishing points but has a constant gradient.

(iv) With what speed do swimmers arrive at the end of this chute? [4 marks]

6 Impulse and momentum

I collided with a stationary truck coming the other way.

Statement on an insurance form reported in the Toronto News

→ The karate expert in the photograph has just broken a pile of wooden planks with a single blow from his hand. Forces in excess of 3000 N have been measured during karate chops. How is this possible?

1 Impulse

Although the karate expert produces a very large force, it acts for only a short time. This is often the case in situations where impacts occur, as in the following example involving a tennis player.

Example 6.1

A tennis player hits the ball as it is travelling towards her at $10\,\mathrm{m\,s^{-1}}$ horizontally. Immediately after she hits it, the ball is travelling away from her at $20\,\mathrm{m\,s^{-1}}$ horizontally. The mass of the ball is 0.06 kg. What force does the tennis player apply to the ball?

Discussion point

Show that the average force she applies to the ball in the cases where the impact lasts 0.1 s and 0.015 s are 18 N and 120 N, respectively. What does 'average' mean in this context?

Solution

You cannot tell unless you know how long the impact lasts, and that will vary from one shot to another.

Although you cannot calculate the force unless you know the time for which it acts, you can work out the product force × time. This is called the *impulse*. An impulse is usually denoted by **J** and its magnitude by J.

When a constant force acts for a time t the impulse of the force is defined as

$$\text{impulse} = \text{force} \times \text{time}.$$

The impulse is a vector in the direction of the force. Impulse is often used when force and time cannot be known separately but their combined effect is known, as in the case of the tennis ball. The S.I. unit for impulse is the newton second (N s).

Discussion point

The magnitude of the momentum of an object is often thought of as its resistance to being stopped. Compare the momentum and kinetic energy of a cricket ball of mass 0.15 kg bowled very fast at $40\,\mathrm{m\,s^{-1}}$ and a 20 tonne railway truck moving at the very slow speed of 1 cm per second.

Which would you rather be hit by, an object with high momentum and low energy, or one with high energy and low momentum?

Impulse and momentum

When the motion is in one dimension and the velocity of an object of mass m is changed from u to v by a constant force F, you can use Newton's second law and the equations for motion with constant acceleration.

$$F = ma$$

and $$v = u + at$$

$$\Rightarrow \quad mv = mu + mat$$

Substituting F for ma gives $$mv = mu + Ft$$

$$\Rightarrow \quad Ft = mv - mu \qquad ①$$

The quantity 'mass × velocity' is defined as the *momentum* of the moving object.

The equation ① can be written as

$$\text{impulse of force} = \text{final momentum} - \text{initial momentum} \qquad ②$$

So impulse = change in momentum

This equation also holds for any large force acting for a short time even when it cannot be assumed to be constant. The force on the tennis ball will increase as it embeds itself into the strings and then decrease as it is catapulted away, but you can calculate the impulse of the tennis racket on the ball as

$$0.06 \times 20 - 0.06 \times (-10) = 1.8\,\mathrm{N\,s}$$

final momentum

initial momentum

the −10 takes account of the change in direction

impulse

Equation ② is also true for a variable force. It is also true, but less often used, when a longer time is involved.

Example 6.2

A ball of mass 50 g hits the ground with a speed of $4\,\mathrm{m\,s^{-1}}$ and rebounds with an initial speed of $3\,\mathrm{m\,s^{-1}}$. The situation is modelled by assuming that the ball is in contact with the ground for 0.01 s and that during this time the reaction force on it is constant.

(i) Find the average force exerted on the ball.

(ii) Find the loss in kinetic energy during the impact.

(iii) Which of the answers to parts (i) and (ii) would be affected by a change in the modelling assumption that the ball is only in contact with the ground for 0.01 s?

Solution

(i) The impulse is given by:

$$J = mv - mu$$
$$= 0.05 \times 3 - 0.05 \times (-4)$$
$$= 0.35$$

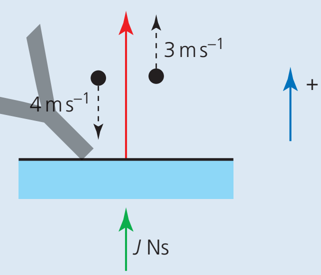


Figure 6.1

The impulse is also given by

$$J = Ft$$

where F is the average force, i.e. the constant force which, acting for the same time interval, would have the same effect as the variable force which actually acted.

$$\therefore \quad 0.35 = F \times 0.01$$
$$F = 35$$

So the ground exerts an average upward force of 35 N.

(ii) Initial K.E. $= \frac{1}{2} \times 0.05 \times 4^2$
$= 0.400$ joules

Final K.E. $= \frac{1}{2} \times 0.05 \times 3^2$
$= 0.225$ joules

Loss in K.E. = 0.175 joules

(This is converted into heat and sound.)

(iii) A change in the model will affect the answer to part (i), but not part (ii).

Note

Example 6.2 demonstrates the important point that mechanical energy is not conserved during an impact.

Although the force of gravity acts during the impact, its impulse is negligible over such a short time.

Example 6.3

A car of mass 800 kg is pushed with a constant force of magnitude 200 N for 10 s. The car starts from rest. Resistance to motion may be ignored.

(i) Find its speed at the end of the ten-second interval by using

(a) the impulse on the car (b) Newton's second law.

(ii) Comment on your answers to part (i).

Solution

The impulse is in the direction of the force.

(i) (a) The force of 200 N acts for 10 s, so the impulse on the car is

$$J = 200 \times 10 = 2000\,\text{N s}$$

Hence the change in momentum (in N s) is

$$mv = 2000$$

$$\therefore \qquad v = \frac{2000}{800} = 2.5$$

The speed at the end of the time interval is $2.5\,\text{m s}^{-1}$.

Since the force is assumed to be a constant, so is the acceleration and so you can use the constant acceleration formulae.

(b) Newton's second law

$$F = ma$$

$$200 = 800a$$

$$a = 0.25\ \text{m s}^{-2}$$

$$v = u + at$$

$$v = 0 + 0.25 \times 10 = 2.5\ \text{m s}^{-1}$$

(ii) Both methods give the same answer but the method based on Newton's second law and the constant acceleration formulae only works because the force is constant.

Consider a variable force $F(t)$ acting on an object in the interval of time $0 \le t \le T$ which changes its velocity from U to V.

At any instant, Newton's second law gives

$$F = ma = m\,\frac{dv}{dt}$$

and so the overall effect is given by

$$\int_0^T F\,\mathrm{d}t = m\int_{v=U}^{v=V} \frac{\mathrm{d}v}{\mathrm{d}t}\,\mathrm{d}t = m\int_U^V \mathrm{d}v$$

$$= mV - mU$$

This is the impulse–momentum equation.

Exercise 6.1

① Find the momentum of the following objects, assuming each of them to be travelling in a straight line.

(i) An ice skater of mass 50 kg travelling with speed $10\,\text{m s}^{-1}$.

(ii) An elephant of mass 5 tonnes moving at $4\,\text{m s}^{-1}$.

(iii) A train of mass 7000 tonnes travelling at $40\,\text{m s}^{-1}$.

(iv) A bacterium of mass 2×10^{-16} g moving with speed $1\,\text{mm s}^{-1}$.

② Calculate the impulse required in each of these situations:

(i) to stop a car of mass 1.3 tonnes travelling at $14\,\text{m s}^{-1}$

(ii) to putt a golf ball of mass 1.5 g with speed $1.5\,\text{m s}^{-1}$

(iii) to stop a cricket ball of mass 0.15 kg travelling at $20\,\text{m s}^{-1}$

(iv) to fire a bullet of mass 25 g with speed $400\,\text{m s}^{-1}$.

③ A stone of mass 1.5 kg is dropped from rest. After a time interval t s, it has fallen a distance s m and has velocity $v\,\mathrm{m\,s^{-1}}$.

Take g to be $10\,\mathrm{m\,s^{-2}}$ and neglect air resistance.

(i) Write down the force F (in N) acting on the stone.

(ii) Find s when $t = 2$.

(iii) Find v when $t = 2$.

(iv) Write down the value, units and meaning of Fs and explain why this has the same value as $\frac{1}{2} \times 1.5v^2$.

(v) Write down the value, units and meaning of Ft and explain why this has the same value as $1.5v$.

④ A ball of mass 200 g is moving in a straight line with a speed of $5\,\mathrm{m\,s^{-1}}$ when a force of 20 N is applied to it for 0.1 s in the direction of motion. Find the final speed of the ball

(i) using the impulse–momentum equation

(ii) using Newton's second law and the constant acceleration formulae.

(iii) Compare the methods.

⑤ A girl throws a ball of mass 0.06 kg vertically upwards with initial speed $20\,\mathrm{m\,s^{-1}}$.

Take g to be $10\,\mathrm{m\,s^{-2}}$ and neglect air resistance.

(i) What is the initial momentum of the ball?

(ii) How long does it take for the ball to reach the top of its flight?

(iii) What is the momentum of the ball when it is at the top of its flight?

(iv) What impulse acted on the ball over the period between its being thrown and its reaching maximum height?

⑥ A netball of mass 425 g is moving horizontally with speed $5\,\mathrm{m\,s^{-1}}$ when it is caught.

(i) Find the impulse needed to stop the ball.

(ii) Find the average force needed to stop the ball if it takes

(a) 0.1 s (b) 0.05 s.

(iii) Why does the action of taking a ball into your body make it easier to catch?

⑦ A car of mass 0.9 tonnes is travelling at $13.2\,\mathrm{m\,s^{-1}}$ when it crashes head-on into a wall. The car is brought to rest in a time of 0.12 s. Taking g to be $10\,\mathrm{m\,s^{-2}}$, find

(i) the impulse acting on the car

(ii) the average force acting on the car

(iii) the average deceleration of the car in terms of g.

(iv) Explain why many cars are designed with crumple zones rather than with completely rigid construction.

⑧ Boris is sleeping on a bunk-bed at a height of 1.5 m when he rolls over and falls out. His mass is 20 kg.

(i) Find the speed with which he hits the floor.

(ii) Find the impulse that the floor has exerted on him when he has come to rest.

(iii) Find the impulse he has exerted on the floor.

It takes Boris 0.2 s to come to rest.

(iv) Find the average force acting on him during this time.

⑨ A railway truck of mass 10 tonnes is travelling at $3\,\text{m}\,\text{s}^{-1}$ along a siding when it hits some buffers. After the impact it is travelling at $1.5\,\text{m}\,\text{s}^{-1}$ in the opposite direction.

(i) Find the initial momentum of the truck.

(ii) Find the momentum of the truck after it has left the buffers.

(iii) Find the impulse that has acted on the truck.

During the impact the force F N that the buffers exert on the truck varies as shown in this graph.

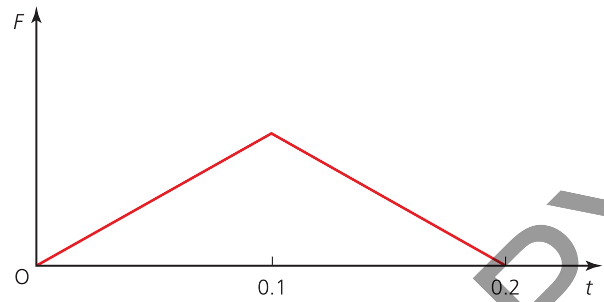


Figure 6.2

(iv) State what information is given by the area under the graph.

(v) What is the greatest value of the force F.

⑩ A van of mass 2500 kg starts from rest. In the first 4 seconds after starting, the driving force on its engine follows the relationship $F(t) = 2400t - 300t^2$.

(i) Find the total impulse on the van over the 4 seconds.

(ii) Find the speed of the van, ignoring the effect of air resistance.

2 Conservation of momentum

Collisions

In an experiment to investigate car design, two vehicles were made to collide head-on. How would you investigate this situation? What is the relationship between the change in momentum of the van and that of the car?

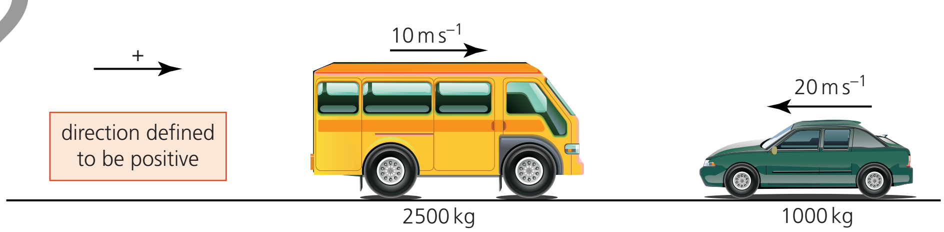


Figure 6.3

Remember Newton's third law. The force that body A exerts on body B is equal to the force that B exerts on A, but in the opposite direction.

Suppose that once the van is in contact with the car, it exerts a force F on the car for a time t. Newton's third law tells us that the car also exerts a force F on the van for a time t. (This applies whether F is constant or variable.) So both vehicles receive equal impulses, but in opposite directions. Consequently, the increase in momentum of the car in the positive direction is exactly equal to the increase in momentum of the van in the negative direction. For the two vehicles together, the total change in momentum is zero.

This example illustrates the *law of conservation of momentum*.

The law of conservation of momentum states that when there are no external influences on a system, the total momentum of the system is constant.

It is important to remember that although momentum is conserved in a collision, mechanical energy is not conserved. Some of the work done by the forces is converted into heat and sound.

Since momentum is a vector quantity, this applies to the magnitude of the momentum in any direction.

For a collision you can say

total momentum before collision = total momentum after collision

Example 6.4

The two vehicles in the previous discussion collide head-on, and, as a result, the van comes to rest.

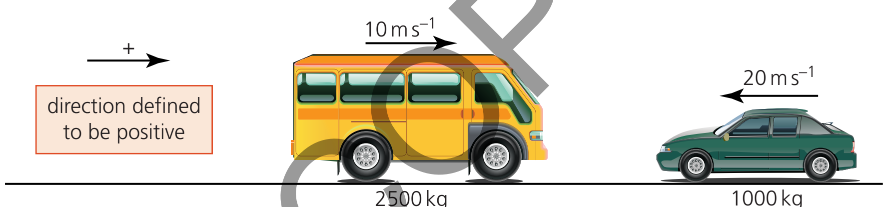


Figure 6.4

(i) Draw diagrams showing the situation before and after the collision.

(ii) Find the final velocity of the car, $v\,\text{m}\,\text{s}^{-1}$

(iii) Find the impulse on each vehicle

(iv) Find the kinetic energy lost.

(v) In modelling the collision, it is assumed that the impact lasts for one-twentieth of a second. Find the average force on each vehicle and the acceleration of each vehicle.

Solution

(i) The vehicles are modelled as particles.

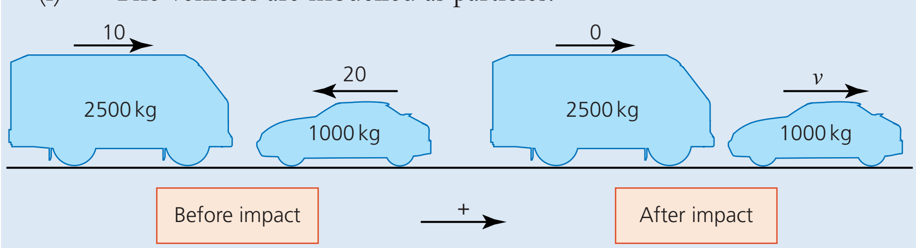


Figure 6.5

(ii) Using conservation of momentum, and taking the positive direction as being to the right:

$$2500 \times 10 + 1000 \times (-20) = 2500 \times 0 + 1000 \times v$$

$$5000 = 1000\,v$$

$$v = 5$$

Note

When you are solving an impact problem, always draw a 'before' and 'after' diagram like this one.

All relevant information on the masses and velocities of the two vehicles is given on it.

Note

The vehicles experience equal and opposite impulses.

The final velocity of the car is $5\,\mathrm{m\,s^{-1}}$ in the positive direction (i.e. the car travels backwards).

(iii) Impulse = final momentum − initial momentum

For the van, impulse $= 2500 \times 0 - 2500 \times 10$

$= -25\,000$ N s

For the car, impulse $= 1000 \times 5 - 1000 \times (-20)$

$= +25\,000$ N s

The van experiences an impulse of 25 000 N s in the negative direction, the car 25 000 N s in the positive direction.

(iv) $$\text{Total initial K.E.} = \frac{1}{2}\times 2500 \times 10^2 + \frac{1}{2}\times 1000 \times 20^2$$

$$= 325\,000 \text{ joules}$$

$$\text{Total final K.E.} = \frac{1}{2}\times 2500 \times 0^2 + \frac{1}{2}\times 1000 \times 5^2$$

$$= 12\,500 \text{ joules}$$

$$\text{Loss in K.E.} = 312\,500 \text{ joules}$$

(v) The impulse is equal to the average force × time. If F is the average force, then

$$25\,000 = F \times \frac{1}{20}$$

$$F = 500\,000\ \mathrm{N}$$

The average force on the car is 500 000 N to the right and that on the van is 500 000 N to the left.

Using $F = ma$ on each vehicle gives an average acceleration of $500\,\mathrm{m\,s^{-2}}$ for the car and $-200\,\mathrm{m\,s^{-2}}$ for the van.

This is over 50g and most people black out at less than 10g.

Hint

Be careful not to confuse J as a symbol for impulse and as the short form of joule, the unit for energy.

Discussion point

These accelerations ($500\,\mathrm{m\,s^{-2}}$ and $-200\,\mathrm{m\,s^{-2}}$) seem very high. Are they realistic for a head-on collision?

Work out the distance each car travels during the time interval of one-twentieth of a second between impact and separation. This will give you an idea of the amount of damage there would be.

Is it better for cars to be made strong so that there is little damage, or to be designed to crumple under impact?

Example 6.5

In an experiment on lorry bumper design, the Transport Research Laboratory arranged for a car and a lorry, of masses 1 and 3.5 tonnes, to travel towards each other, both with speed $9\,\mathrm{m\,s^{-1}}$.

After colliding, both vehicles moved together. What was their combined velocity after the collision?

Solution

The situation before the collision is illustrated below.

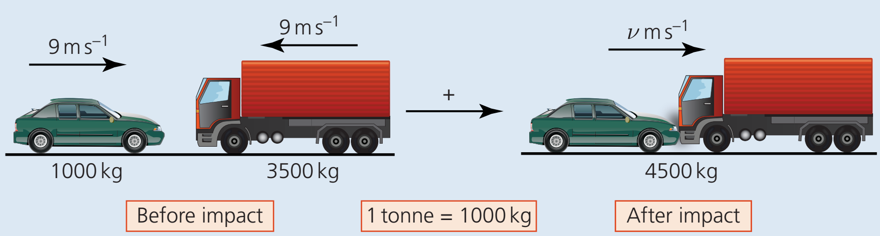


Figure 6.6

Hint

The vehicles are treated as particles and all relevant information is in the diagram.

Taking the positive direction to be to the right, before the collision

momentum of the car in N s: $1000 \times 9 = 9000$

momentum of the lorry in N s: $3500 \times (-9) = -31\ 500$

total momentum in N s: $9000 - 31\ 500 = -22\ 500$

After the collision, assume that they move as a single object of mass 4.5 tonnes with velocity $v\,\text{m s}^{-1}$ in the positive direction so that the total momentum is now $4500v$ N s.

Momentum is conserved so $4500v = -22\ 500$

$$v = -5$$

The car and lorry move at $5\,\text{m s}^{-1}$ in the direction in which the lorry was moving.

Example 6.6

A child of mass 30 kg running through a supermarket at $4\,\text{m s}^{-1}$ leaps on to a stationary shopping trolley of mass 15 kg. Find the speed of the child and trolley together, assuming that the trolley is free to move easily.

Solution

The diagram shows the situation before the child hits the trolley

Before impact

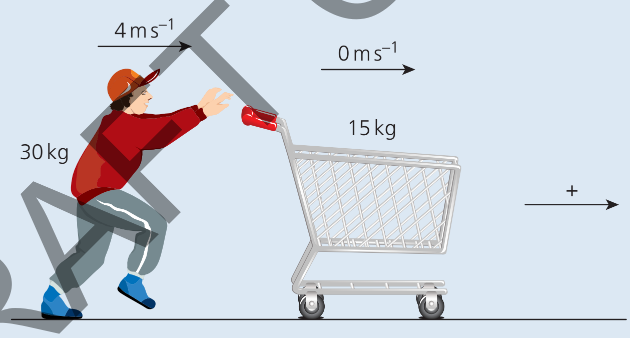


Figure 6.7

Taking the direction of the child's velocity as positive, the total momentum before impact is equal to $4 \times 30 + 0 \times 15 = 120$ N s.

The situation after impact is shown below.

After impact

Figure 6.8

The total mass of child and trolley is 45 kg, so the total momentum after is $45v$ N s.

Conservation of momentum gives:

$45v = 120$

$v = 2\frac{2}{3}$

The child and the trolley together move at $2\frac{2}{3}\,\mathrm{m\,s^{-1}}$.

Explosions

Conservation of momentum also applies when explosions take place provided there are no external forces. For example when a bullet is fired from a rifle, or a rocket is launched.

Example 6.7

A rifle of mass 8 kg is used to fire a bullet of mass 80 g at a speed of $200\,\mathrm{m\,s^{-1}}$. Calculate the initial recoil speed of the rifle.

Solution

Before the bullet is fired, the total momentum of the system is zero.

Before explosion

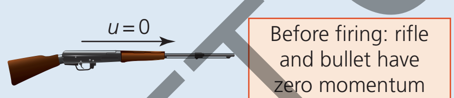


Figure 6.9

After firing, the situation is as illustrated below.

After explosion

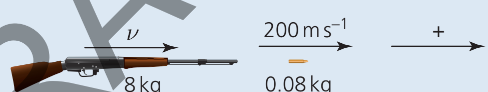


Figure 6.10

The total momentum in the positive direction after the firing is $8v + 0.08 \times 200$.

For momentum to be conserved,

$$8v + 0.08 \times 200 = 0$$

so that

$$v = \frac{-0.08 \times 200}{8} = -2$$

You have probably realised that v would turn out to be negative.

The recoil speed of the rifle is $2\,\mathrm{m\,s^{-1}}$.

Exercise 6.2

① A spaceship of mass 50 000 kg travelling with speed 200 m s^{-1} docks with a space station of mass 500 000 kg travelling in the same direction with speed 195 m s^{-1}. What is their speed after the docking is completed?

② A railway truck of mass 20 tonnes is shunted with speed 3 m s^{-1} towards a stationary truck of mass 10 tonnes. What is the speed after impact

(i) if the two trucks remain in contact?

(ii) if the second truck now moves at 3 m s^{-1}?

③ The driver of a car of mass 1000 kg falls asleep while it is travelling at 30 m s^{-1}. The car runs into the back of the car in front which has mass 800 kg and is travelling in the same direction at 20 m s^{-1}. The bumpers of the two cars become locked together and they continue as one vehicle.

(i) What is the speed of the cars immediately after impact?

(ii) What impulse does the larger car give to the smaller one?

(iii) What impulse does the smaller car give to the larger one?

④ A lorry of mass 5 tonnes is about to tow a car of mass 1 tonne. Initially the tow rope is slack and the car stationary. As the rope becomes taut the lorry is travelling at 2 m s^{-1}.

(i) Find the speed of the car once it is being towed.

(ii) Find the magnitude of the impulse transmitted by the tow rope and state the direction of the impulse on each vehicle.

⑤ A bullet of mass 50 g is moving horizontally at 200 m s^{-1} when it becomes embedded in a stationary block of mass 16 kg which is free to slide on a smooth horizontal table.

(i) Calculate the speed of the bullet and the block after the impact.

(ii) Find the impulse from the bullet on the block.

The bullet takes 0.01 s to come to rest relative to the block.

(iii) What is the average force acting on the bullet while it is decelerating?

⑥ A spaceship of mass 50 000 kg is travelling through space with speed 5000 m s^{-1} when a crew member throws a box of mass 5 kg out of the back with speed 10 m s^{-1} relative to the spaceship.

(i) What is the absolute speed of the box?

(ii) What is the speed of the spaceship after the box has been thrown out?

⑦ A gun of mass 500 kg fires a shell of mass 5 kg horizontally with muzzle speed 300 m s^{-1}.

(i) Calculate the recoil speed of the gun.

An army commander would like soldiers to be able to fire such a shell from a rifle held against their shoulders (so they can attack armoured vehicles).

(ii) Explain why such an idea has no hope of success.

⑧ Manoj (mass 70 kg) and Alka (mass 50 kg) are standing stationary facing each other on a smooth ice rink. They then push against each other with a force of 35 N for 1.5 s. The direction in which Manoj faces is taken as positive.

(i) What is their total momentum before they start pushing?

(ii) Find the velocity of each of them after they have finished pushing.

(iii) Find the momentum of each of them after they have finished pushing.

(iv) What is their total momentum after they have finished pushing?

⑨ A truck P of mass 2000 kg starts from rest and moves down an incline from A to B as illustrated in the diagram. The distance from A to B is 50 m and $\sin\alpha = 0.05$. CBDE is horizontal.

Figure 6.11

Neglecting resistance to motion, calculate:

(i) the potential energy lost by the truck P as it moves from A to B

(ii) the speed of the truck P at B.

Truck P then continues from B without loss of speed towards a second truck Q of mass 1500 kg at rest at D. The two trucks collide and move on towards E together. Still neglecting resistances to motion, calculate:

(iii) the common speed of the two trucks just after they become coupled together

(iv) the percentage loss of kinetic energy in the collision.

⑩ Katherine (mass 40 kg) and Elizabeth (mass 30 kg) are on a sledge (mass 10 kg) which is travelling across smooth horizontal ice at $5\,\text{m}\,\text{s}^{-1}$. Katherine jumps off the back of the sledge with speed $4\,\text{m}\,\text{s}^{-1}$ backwards relative to the sledge.

(i) What is Katherine's absolute speed when she jumps off?

(ii) With what speed does Elizabeth, still on the sledge, then go?

Elizabeth then jumps off in the same manner, also with speed $4\,\text{m}\,\text{s}^{-1}$ relative to the sledge.

(iii) What is the speed of the sledge now?

(iv) What would the final speed of the sledge have been if Katherine and Elizabeth had both jumped off at the same time, with speed $4\,\text{m}\,\text{s}^{-1}$ backwards relative to the sledge?

3 Newton's law of impact

Discussion point

If you drop two different balls, say a tennis ball and a cricket ball, from the same height, will they both rebound to the same height? How will the heights of the second bounces compare with the heights of the first ones?

Your own experience probably tells you that different balls will rebound to different heights.

For example, a tennis ball will rebound to a greater height than a cricket ball. Furthermore, the surface on which the ball is dropped will affect the bounce. A tennis ball dropped onto a concrete floor will rebound higher than if dropped onto a carpeted floor. The following experiment allows you to look at this situation more closely.

EXPERIMENT

The aim of this experiment is to investigate what happens when balls bounce. Make out a table to record your results.

1. Drop a ball from a variety of heights and record the heights of release h_a and rebound h_s. Repeat several times for each height.
2. Use your values of h_a and h_s, to calculate v_a and v_s, the speeds on impact and rebound. Enter your results in your table.
3. Calculate the ratio $\frac{v_a}{v_s}$ for each pair of readings of h_a and h_s and enter the results in your table.
4. What do you notice about these ratios?
5. Repeat the experiment with different types of ball.

Coefficient of restitution

Newton's experiments on collisions led him to formulate a simple law relating to the speeds before and after a direct collision between two bodies, called *Newton's law of impact*.

$$\frac{\text{speed of separation}}{\text{speed of approach}} = \text{constant}$$

This can be written as

$$\text{speed of separation} = \text{constant} \times \text{speed of approach}$$

This constant is called the *coefficient of restitution* and is conventionally denoted by the letter e. For two particular surfaces, e is a constant between 0 and 1. It does not have any units, being the ratio of two speeds.

For very bouncy balls, e is close to 1, and for balls that do not bounce, e is close to 0. A collision for which $e = 1$ is called perfectly elastic, and a collision for which $e = 0$ is called perfectly inelastic.

For perfectly elastic collisions there is no energy loss. For perfectly inelastic collisions the objects coalesce and the energy loss is the largest it can be.

Direct impact with a fixed surface

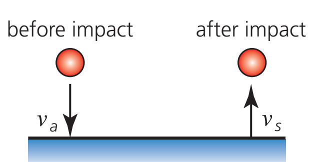


Figure 6.12

The value of e for the ball you used in the experiment is given by $\frac{v_s}{v_a}$, and you should have found that this had approximately the same value each time for any particular ball. When a moving object hits a fixed surface which is perpendicular to its motion, it rebounds in the opposite direction. If the speed of approach is v_a and the speed of separation is v_s,

Newton's law of impact gives

$$\frac{v_s}{v_a} = e$$

$$\Rightarrow \qquad v_s = ev_a$$

Collisions between bodies moving in the same straight line

Figure 6.13 shows two objects that collide while moving along a straight line. Object A is catching up with B, and after the collision either B moves away from A or they continue together.

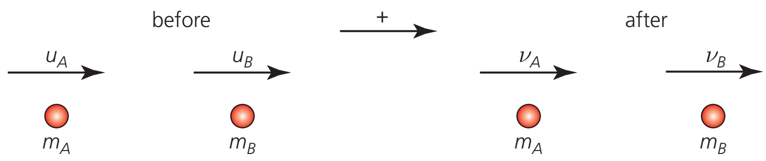


Figure 6.13

$u_A > u_B$ for the collision to occur

$v_B \geq v_A$ as B moves away from A

If the particles coalesce then $v_A = v_B$

Speed of approach: $u_A - u_B$

Speed of separation: $v_B - v_A$

By Newton's law

$$\text{speed of separation} = e \times \text{speed of approach}$$

$$\Rightarrow \quad v_B - v_A = e(u_A - u_B) \qquad (1)$$

The law of conservation of momentum gives a second equation relating the velocities before and after impact.

$$\text{momentum after collision} = \text{momentum before collision}$$

$$m_A v_A + m_B v_B = m_A u_A + m_B u_B \qquad (2)$$

These two equations (1) and (2), allow you to calculate the final velocities, v_A and v_B, after any collision as shown in the next two examples.

Example 6.8

A direct collision takes place between two snooker balls. The white cue ball travelling at $2\,\text{m s}^{-1}$ hits a stationary red ball. After the collision, the red ball moves in the direction in which the cue ball was moving before the collision. The balls have equal mass and the coefficient of restitution between the two balls is 0.6. Predict the velocities of the two balls after the collision.

Solution

Let the mass of each ball be m. Before the collision, their velocities are u_W and u_R. After the collision, their velocities are v_W and v_R.

The situation is summarised in Figure 6.14.

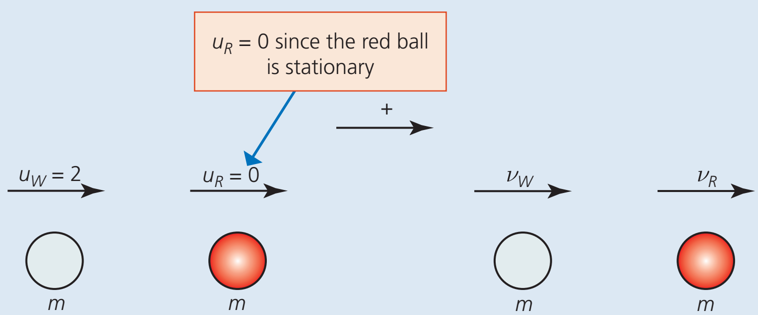


Figure 6.14

By Newton's law of impact

Speed of approach = 2 − 0 = 2

Speed of separation = $v_R - v_W$

Speed of separation = $e \times$ Speed of approach

$\Rightarrow \quad v_R - v_W = 0.6 \times 2$

$\Rightarrow \quad v_R - v_W = 1.2$ ①

Conservation of momentum

$$mv_W + mv_R = mu_W + mu_R$$

Dividing through by m, and substituting $u_W = 2$, $u_R = 0$, this becomes

$$v_W + v_R = 2 \quad ②$$

Adding ① + ② gives $2v_R = 3.2$

so $v_R = 1.6$, and from equation ②, $v_W = 0.4$.

After the collision both balls move in the original direction of the white cue ball, the red ball at a speed of $1.6\,\text{m s}^{-1}$ and the cue ball at a speed of $0.4\,\text{m s}^{-1}$.

Example 6.9

An object A of mass m moving with speed $2u$ hits an object B of mass $2m$ moving with speed u in the opposite direction to A. The coefficient of restitution is e.

(i) Show that the ratio of speeds remains unchanged whatever the value of e.

(ii) Find the loss of kinetic energy in terms of m, u and e.

Solution

(i) Let the velocities of A and B after the collision be v_A and v_B, respectively.

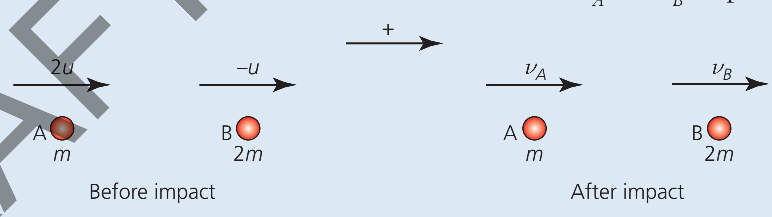


Figure 6.15

$$\text{Speed of approach} = 2u - (-u) = 3u$$

$$\text{Speed of separation} = v_B - v_A$$

Using Newton's law of impact

$$\text{speed of separation} = e \times \text{speed of approach}$$

$$v_B - v_A = e \times 3u \quad ①$$

Conservation of momentum gives

$$mv_A + 2mv_B = m(2u) + 2m(-u)$$

Dividing by m gives

$$v_A + 2v_B = 0 \quad ②$$

Equation ① is $v_B - v_A = 3eu$

Adding ① + ② $3v_B = 3eu$

$v_B = eu$

From ②, $v_A = -2eu$

The ratio of speeds was initially $2u : u$ and finally $2eu : eu$ so the ratio of speeds is unchanged at 2 : 1 (providing $e \neq 0$).

(ii)

Initial K.E. of A $\frac{1}{2}m \times (2u)^2 = 2mu^2$

Initial K.E. of B $\frac{1}{2}(2m) \times u^2 = mu^2$

Total K.E before impact $= 3mu^2$

Final K.E. of A $\frac{1}{2}m \times 4e^2u^2 = 2me^2u^2$

Final K.E. of B $\frac{1}{2}(2m) \times e^2u^2 = me^2u^2$

Total K.E. after impact $= 3me^2u^2$

Loss of K.E. $= 3mu^2(1 - e^2)$

Note

In this case, A and B lose *all* their energy when $e = 0$, but this is not true in general. Only when $e = 1$ is there no loss in K.E. Kinetic energy is lost in any collision in which the coefficient of restitution is not equal to 1.

Exercise 6.3

You will find it helpful to draw diagrams when answering these questions.

① In each of the situations shown below, find the unknown quantity, either the initial speed u, the final speed v or the coefficient of restitution e.

② Find the coefficient of restitution in the following situations.

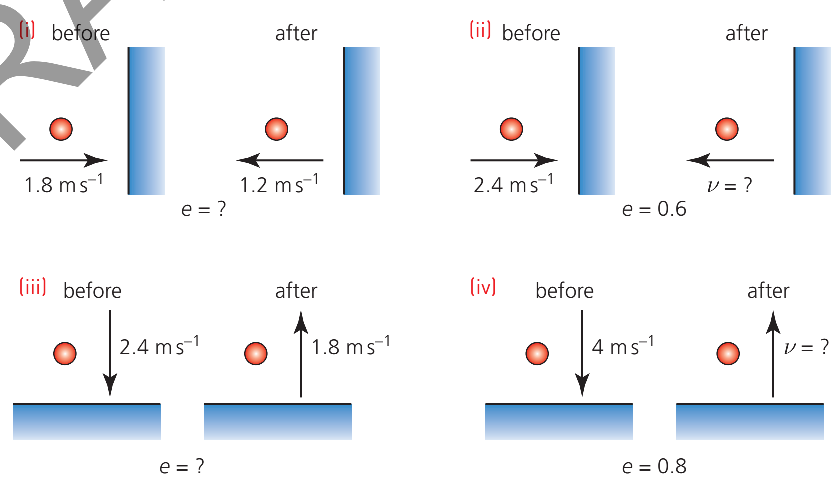


Figure 6.16

(i) A football hits the goalpost at $10\,\text{m s}^{-1}$ and rebounds in the opposite direction with speed $3\,\text{m s}^{-1}$.

(ii) A beanbag is thrown against a wall with speed $5\,\text{m s}^{-1}$ and falls straight down to the ground.

(iii) A superball is dropped onto the ground, landing with speed $8\,\text{m s}^{-1}$ and rebounds with speed $7.6\,\text{m s}^{-1}$.

(iv) A photon approaches a mirror along a line normal to its surface with speed $3 \times 10^8\,\text{m s}^{-1}$ and leaves it along the same line with speed $3 \times 10^8\,\text{m s}^{-1}$.

③ A tennis ball of mass 60 g is hit against a practice wall. At the moment of impact it is travelling horizontally with speed $15\,\text{m s}^{-1}$. Just after the impact its speed is $12\,\text{m s}^{-1}$, also horizontally. Find:

(i) the coefficient of restitution between the ball and the wall

(ii) the impulse acting on the ball

(iii) the loss of kinetic energy during the impact.

④ A ball of mass 80 g is dropped from a height of 1 m onto a level floor and bounces back to a height of 0.81 m. Find:

(i) the speed of the ball just before it hits the floor

(ii) the speed of the ball just after it has hit the floor

(iii) the coefficient of restitution

(iv) the change in the kinetic energy of the ball from just before it hits the floor to just after it leaves the floor

(v) the change in the potential energy of the ball from the moment when it was dropped to the moment when it reaches the top of its first bounce

(vi) the height of the ball's next bounce.

⑤ In each of the situations below, a collision is about to occur. Masses are given in kilograms, speeds are in metres per second. In each case

(i) draw diagrams showing the situation before and after impact, including known velocities and the symbols you are using for velocities that are not yet known.

(ii) Use the equations corresponding to the law of conservation of momentum and to Newton's law of impact to find the final velocities.

(iii) Find the loss of kinetic energy during the collision.

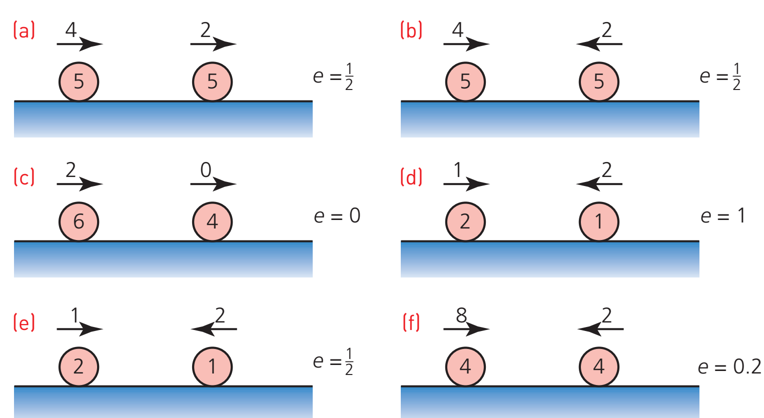


Figure 6.17

6 An object of mass 10 kg is acted on by a force whose magnitude varies according to the distance from the starting point O, as shown on the graph. The force acts in a constant direction.

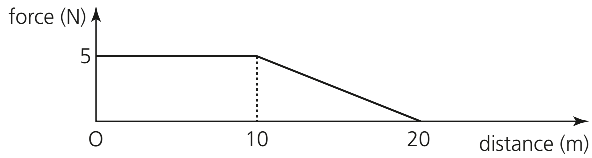


Figure 6.18

(i) What work has been done by the force when the object reaches the point 20 m from O?

(ii) If the object starts from rest, what is its final speed?

(iii) What is its final momentum?

(iv) What is the total impulse of the force over the period?

7 Two children drive dodgems straight at each other, and collide head-on. Both dodgems have the same mass (including their drivers) of 150 kg. Isobel is driving at $3\,\mathrm{m\,s^{-1}}$, Stuart at $2\,\mathrm{m\,s^{-1}}$. After the collision Isobel is stationary. Find

(i) Stuart's velocity after the collision

(ii) the coefficient of restitution between the cars

(iii) the impulse acting on Stuart's car

(iv) the kinetic energy lost in the collision.

8 A trapeze artist of mass 50 kg falls from a height of 20 m into a safety net.

(i) Find the speed with which she hits the net. (You may ignore air resistance.)

Her speed on leaving the net is $15\,\mathrm{m\,s^{-1}}$.

(ii) What is the coefficient of restitution between her and the net?

(iii) What impulse does the trapeze artist receive?

(iv) How much mechanical energy is absorbed in the impact?

(v) If you were a trapeze artist would you prefer a safety net with a high coefficient of restitution or a low one?

9 Two spheres of equal mass, m, are travelling towards each other along the same straight line when they collide. Both have speed v just before the collision and the coefficient of restitution between them is e. Your answers should be given in terms of m, v and e.

(i) Draw diagrams to show the situation before and after the collision.

(ii) Find the velocities of the spheres after the collision.

(iii) Show that the kinetic energy lost in the collision is given by $mv^2(1 - e^2)$.

(iv) Explain why the result in part (iii) shows that e cannot have a value greater than 1.

⑩ Three identical spheres are lying in the same straight line. The coefficient of restitution between any pair of spheres is $\frac{1}{2}$. Initially the left-hand sphere has a velocity of $2\,\mathrm{m\,s^{-1}}$ towards the other two which are both stationary. What are the final velocities of all three, when no more collisions can occur?

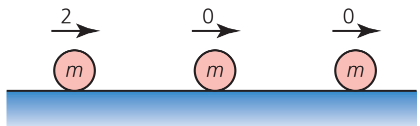


Figure 6.19

⑪ Figure 6.20 shows two snooker balls and one edge cushion. The coefficient of restitution between the balls and the cushion is 0.5 and that between the balls is 0.75. Ball A (the cue ball) is hit directly towards the stationary ball B with speed $8\,\mathrm{m\,s^{-1}}$. Find the speed and directions of the two balls after their second impact with each other.

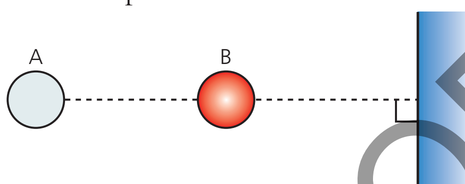


Figure 6.20

⑫ An object of mass 10 kg begins at rest at O (see diagram Figure 6.21), and is acted on by a force $F = 5 - \frac{s^2}{80}$, where s is the distance from O.

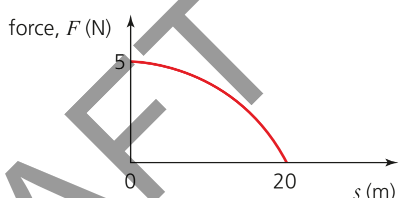


Figure 6.21

(i) What is the speed when $s = 20$?

(ii) What is the total impulse of the force?

⑬ The coefficient of restitution between a ball and the floor is e. The ball is dropped from a height h. Air resistance may be neglected, and your answers should be given in terms of e, h, g and n, the number of bounces.

(i) Find the time it takes the ball to reach the ground and its speed when it arrives there.

(ii) Find the ball's height at the top of its first bounce.

(iii) Find the height of the ball at the top of its nth bounce.

(iv) Find the time that has elapsed when the ball hits the ground for the second time, and for the nth time.

(v) Show that according to this model the ball comes to rest within a finite time having completed an infinite number of bounces.

(vi) What distance does the ball travel before coming to rest?

Impulse of a variable force

For a constant force, the impulse is force × time for which it acts.

When the force varies, the impulse over a small time interval δt is $F\,\delta t$, and so the total impulse over a period is defined as $\int F\,\delta t$ (see Figure 6.22).

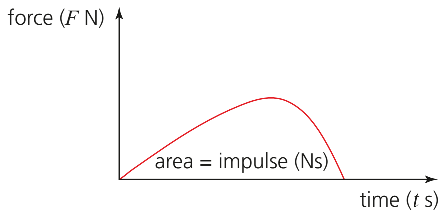


Figure 6.22

As you saw earlier in this chapter, over a period T when the velocity changes from U to V, $\int_0^T F\,dt = mV - mU$, that is, the impulse of a force is equal to the change in momentum. This applies even if the force varies.

Any collision involves variable forces. When two snooker balls, A and B, collide, they are in contact for a very short time. During that time the force between them is not constant. It is zero just as they touch, builds up to a maximum while they deform slightly and then goes down to zero again as the balls rebound. But, by Newton's third law, the force of A and B is equal and opposite to that of B on A at every moment. So, if the total impulse on A is $\int F\ \mathrm{d}t$, that on B is

$$\int -F\ \mathrm{d}t = -\int F\ \mathrm{d}t.$$

The sum of the impulses on A and on B is zero and so the total momentum change is zero. The principle of a conservation of momentum applies even though forces are variable.

4 Impulse and momentum in more than one dimension

Earlier in this chapter you met situations involving impact between objects moving along a straight line. This involved working with momentum and impulse. It is summed up by the equation

final momentum = initial momentum + impulse

These ideas are extended to motion in more than one direction.

Both impulse and momentum are vectors. The impulse of a force is in the direction of the force and the momentum of a moving object is in the direction of its velocity. So the equation above can be represented as the vector diagram in Figure 6.23.

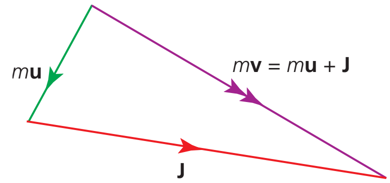


Figure 6.23

Written in symbols, the equation is:

$$m\mathbf{v} = m\mathbf{u} + \mathbf{J}$$

The quantities **u** and **v** can usually be measured but **J** cannot and has to be inferred, using the impulse-momentum equation in the form:

$$\mathbf{J} = m\mathbf{v} - m\mathbf{u}$$

Figure 6.24 shows how this applies to a ball which changes direction when it is hit by a bat.

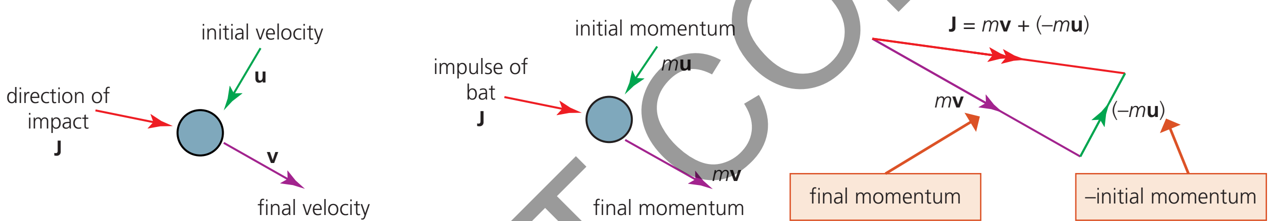


Figure 6.24

Example 6.10

In a game of snooker, the white cue ball of mass 0.2 kg is hit towards a stationary red ball at $0.8\,\text{m s}^{-1}$. After the collision the cue ball is moving at $0.6\,\text{m s}^{-1}$ having been deflected through 30°.

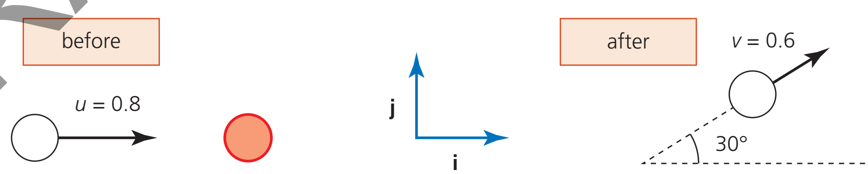


Figure 6.25

Find the impulse on the cue ball and show this in a vector diagram.

Solution

In terms of unit vectors **i** and **j** the velocities of the ball before and after the collision are given by:

$$\mathbf{u} = 0.8\mathbf{i}$$

$$\mathbf{v} = 0.6\cos 30°\,\mathbf{i} + 0.6\sin 30°\,\mathbf{j}$$

Then, using the impulse–momentum equation:

$$\mathbf{J} = m\mathbf{v} - m\mathbf{u}$$

$$= 0.2(0.6\cos 30°\mathbf{i} + 0.6\sin 30°\mathbf{j}) - 0.2(0.8\mathbf{i})$$

$$= -0.056\mathbf{i} + 0.06\mathbf{j}$$

Magnitude of impulse: $= \sqrt{0.056^2 + 0.06^2}$

$= 0.08207...$

Direction of impulse:

$$\tan\alpha = \frac{0.06}{0.056}$$

$$\alpha = 47°$$

$$\theta = 133°$$

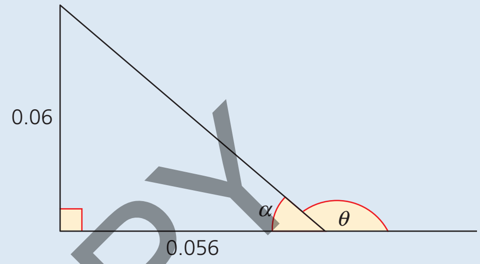


Figure 6.26

The impulse has magnitude 0.082 N s at an angle of 133° to the initial motion of the ball.

This is shown in the vector diagram, Figure 6.27. Note that the impulse–momentum equation shows the direction of the impulsive force acting on the cue ball.

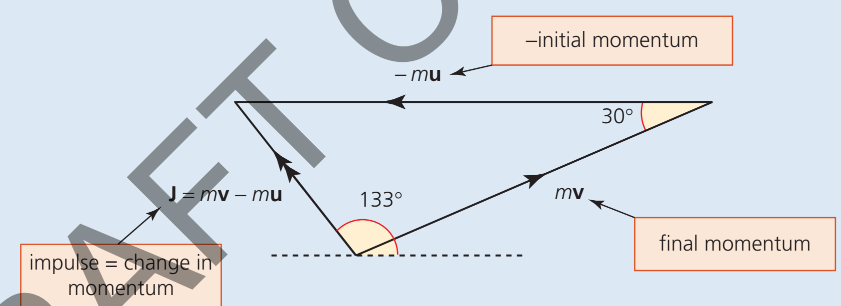


Figure 6.27

Discussion point

What happens to the red ball?

Example 6.11

A hockey ball of mass 0.15 kg is moving at $4\,\mathrm{m\,s^{-1}}$ parallel to the side of a pitch when it is struck by a blow from a hockey stick that exerts an impulse of 4 N s at an angle of 120° to its direction of motion. Find the final velocity of the ball.

Solution

The vector diagram shows the motion of the ball.

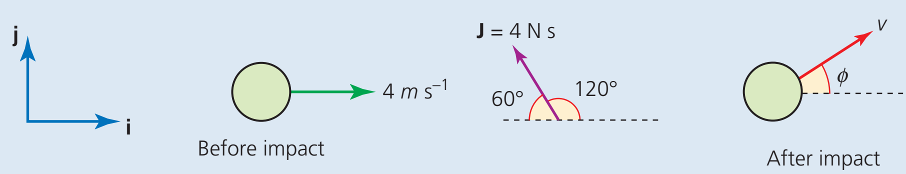


Figure 6.28

In terms of unit vectors **i** and **j**:

$$\mathbf{u} = 4\mathbf{i}$$

and

$$\mathbf{J} = -4\cos 60°\,\mathbf{i} + 4\sin 60°\,\mathbf{j}$$
$$= -2\mathbf{i} + 3.46\mathbf{j}$$

Using

$$\mathbf{J} = m\mathbf{v} - m\mathbf{u}$$
$$-2\mathbf{i} + 3.46\mathbf{j} = 0.15\mathbf{v} - 0.15 \times 4\mathbf{i}$$
$$\Rightarrow \quad 0.15\mathbf{v} = -2\mathbf{i} + 0.6\mathbf{i} + 3.46\mathbf{j}$$
$$\Rightarrow \quad 0.15\mathbf{v} = -1.4\mathbf{i} + 3.46\mathbf{j}$$
$$\Rightarrow \quad \mathbf{v} = -9.33\mathbf{i} + 23.1\mathbf{j}$$

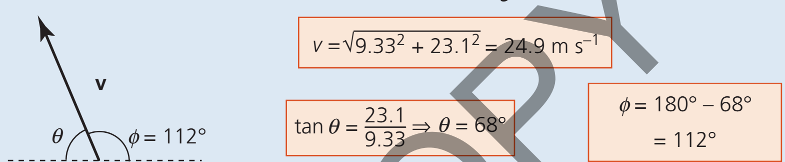


Figure 6.29

After the blow, the ball has a velocity of magnitude 24.9 m s^{-1} at an angle of 112° to the original direction of motion.

If a sphere hits a plane directly (when its velocity is at right angles to the plane) then the mathematics is straightforward. Less straightforward is when a sphere hits a plane at an angle. This becomes an example of a problem that requires resolving a velocity.

! This is A level only content.

Oblique impact of a sphere on a plane

When an object hits a smooth plane there can be no impulse parallel to the plane so the component of momentum, and hence velocity, is unchanged in this direction. Perpendicular to the plane, the momentum is changed but Newton's law of impact still applies.

The diagrams show the components of the velocity of a ball immediately before and after it hits a smooth plane with coefficient of restitution *e*.

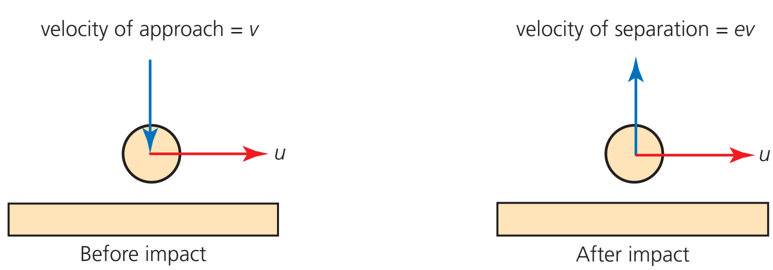


Figure 6.30

When the ball is travelling with speed U at an angle α to the plane, the components of the final velocity are $U \cos \alpha$ parallel to the plane and $eU \sin \alpha$, perpendicular to the plane.

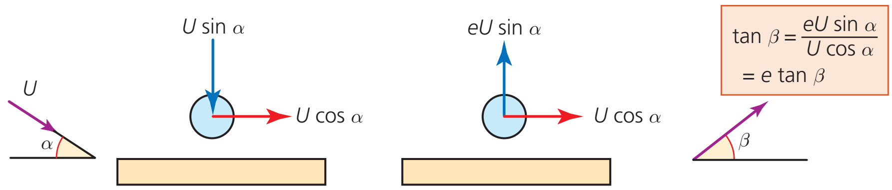


Figure 6.31

The *impulse* on the ball is equal to final momentum – initial momentum. This is perpendicular to the plane because there is no change in momentum parallel to the plane.

In Figure 6.30 the impulse is:

$$mev - m(-v) = (1 + e)mv \text{ upwards.}$$

In Figure 6.31 the impulse is:

$$meU \sin \alpha - m(-U \sin \alpha) = (1 + e)mU \sin \alpha \text{ upwards.}$$

Whenever an impact takes place, energy is likely to be lost. In the cases illustrated in the diagrams, the *loss in kinetic energy* is:

$$\tfrac{1}{2}m(u^2 + v^2) - \tfrac{1}{2}m(u^2 + e^2v^2) = \tfrac{1}{2}m(1 - e^2)v^2$$

or $$\tfrac{1}{2}m(1 - e^2)U^2\sin^2\alpha$$

Discussion point

What happens to the ball when $e = 0$ and $e = 1$?

Example 6.12

A ball of mass 0.2 kg moving at $12\,\text{m s}^{-1}$ hits a smooth horizontal plane at an angle of 75° to the horizontal. The coefficient of restitution is 0.5. Find:

(i) the impulse on the ball

(ii) the impulse on the plane

(iii) the kinetic energy lost by the ball.

Solution

(i) The diagram shows the velocities before and after impact.

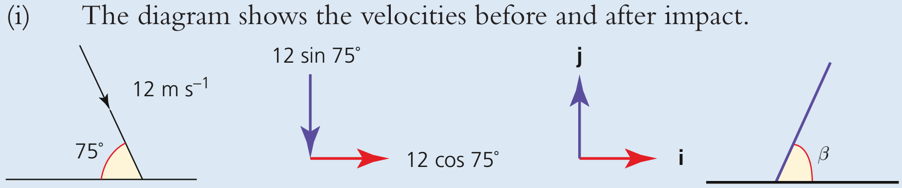


Figure 6.32

Parallel to the plane: $u = 12 \cos 75°$

No change in velocity parallel to the plane.

Perpendicular to the plane: $v = 0.5 \times 12 \sin 75°$

$= 6 \sin 75°$

Using Newton's law of impact with $e = 0.5$.

The impulse on the ball = final momentum − initial momentum

$$\mathbf{J} = 0.2\begin{pmatrix} 12\cos 75° \\ 6\sin 75° \end{pmatrix} - 0.2\begin{pmatrix} 12\cos 75° \\ -12\sin 75° \end{pmatrix}$$

Using directions **i** and **j** as shown.

$$= \begin{pmatrix} 0 \\ 3.6\sin 75° \end{pmatrix}$$

The impulse on the ball is 3.6 sin 75° **j**, that is, 3.48 N s perpendicular to the plane and upwards in the **j** direction.

(ii) By Newton's third law, the impulse on the plane is equal and opposite to the impulse on the ball. It is 3.48 N s perpendicular to the plane in the direction of −**j**.

(iii) The initial kinetic energy $= \frac{1}{2} \times 0.2 \times 12^2 = 14.4$ J

Final kinetic energy $= \frac{1}{2} \times 0.2 \times \left[(12\cos 75°)^2 + (6\sin 75°)^2\right]$

$= 4.32$ J

Loss in kinetic energy $= 14.4 - 4.32 = 10.1$ J (3 s.f.)

Example 6.13

A ball moving with speed $10\,\text{m}\,\text{s}^{-1}$ hits a smooth horizontal plane at an angle of 60° to the horizontal. The coefficient of restitution between the ball and the surface is $\frac{1}{3}$. The ball rebounds with speed v at an angle β with the surface.

(i) Find v.

(ii) Find β.

Solution

(i)

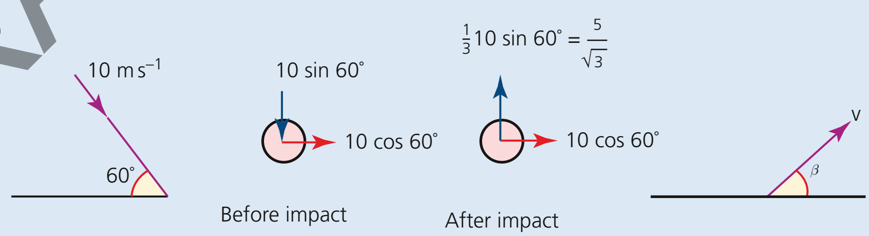


Figure 6.33

$$v = \sqrt{\left(\frac{5}{\sqrt{3}}\right)^2 + 5^2} = \sqrt{\frac{100}{3}} = 5.77\,\text{m}\,\text{s}^{-1}$$

Using $\sin 60° = \frac{\sqrt{3}}{2}$ and $\cos 60° = \frac{1}{2}$.

(ii) $\tan\beta = \frac{\frac{5}{\sqrt{3}}}{5} = \frac{1}{\sqrt{3}}$; $\beta = 30°$

Exercise 6.4

① A snooker ball of mass 0.08 kg is travelling with speed $3.5\,\text{m s}^{-1}$ when it hits the cushion at an angle of 60°. After the impact the ball is travelling with speed $2\,\text{m s}^{-1}$ at an angle of 30° to the cushion.

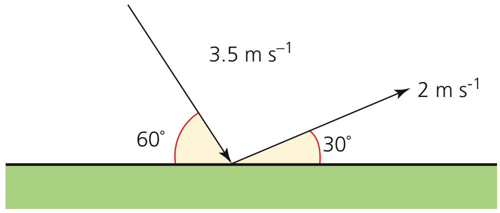


Figure 6.34

(i) Draw accurate scale diagrams to represent the following vectors:
the momentum of the ball before impact
the momentum of the ball after impact
the change in momentum of the ball during impact.

(ii) Use your answers to part (i) to *estimate* the magnitude and direction of the impulse acting on the ball.

(iii) Resolve the velocity of the ball before and after the impact into components parallel and perpendicular to the cushion.

(iv) Use your answers to part (iii) to *calculate* the impulse which acts on the ball during its impact with the cushion. Comment on your answers.

② A hockey ball of mass 0.15 kg is travelling with velocity $12\mathbf{i} - 8\mathbf{j}$ (in m s^{-1}), where the unit vectors **i** and **j** are in horizontal directions parallel and perpendicular to the length of the pitch, and the vector **k** is vertically upwards. The ball is hit by Jane with an impulse $-4.8\mathbf{i} + 1.2\mathbf{j}$.

(i) What is the velocity of the ball immediately after Jane has hit it?

The ball goes straight, without losing any speed, to Fatima in the opposite team who hits it without stopping it. Its velocity is now $14\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$.

(ii) What impulse does Fatima give the ball?

(iii) Which player hits the ball harder?

③ Find the velocity of each of the following after one impact with a smooth plane.

(i) Initial velocity $4\,\text{m s}^{-1}$ at 20° to the plane. Coefficient of restitution 0.5.

(ii) Initial velocity $10\,\text{m s}^{-1}$ at 40° to the plane. Coefficient of restitution 0.1.

(iii) Initial velocity $u\,\text{m s}^{1}$ at α° to the plane. Coefficient of restitution 0.8.

④ A ball of mass 0.1 kg moving at $10\,\text{m s}^{-1}$ hits a smooth horizontal plane at an angle of 80° to the horizontal. The coefficient of restitution is 0.6. Find:

(i) the impulse on the ball

(ii) the impulse on the plane

(iii) the kinetic energy lost by the particle.

⑤ A particle of mass 0.05 kg moving at $8\,\text{m s}^{-1}$ hits a smooth horizontal plane at an angle of 45° to the horizontal. The coefficient of restitution is 0.6. Find:

(i) the impulse on the particle

(ii) the impulse on the plane

(iii) the kinetic energy lost by the particle.

⑥ A ball of mass m kg moving at u m s^{-1} hits a smooth horizontal plane at an angle of $\alpha°$ to the horizontal. The coefficient of restitution is 0.

(i) Find the impulse on the ball.

(ii) Show that the kinetic energy lost is $\frac{1}{2}mu^2\sin^2\alpha$.

⑦ Show that the kinetic energy lost by a particle of mass m kg which hits a smooth plane when it is moving with velocity u m s^{-1} at an angle of $\alpha°$ to the plane is $\frac{1}{2}mu^2\left(1-e^2\right)\sin^2\alpha$, where e is the coefficient of restitution.

⑧ A ball is hit from level ground with initial components of velocity u_x m s^{-1} horizontally and u_y vertically. Assume the ball is a particle and ignore air resistance.

(i) Show that its horizontal range is $R = \frac{2u_x u_y}{g}$.

The ball bounces on the ground with coefficient of restitution 0.6.

(ii) How much further does it travel horizontally before the next bounce?

(iii) Find an expression for the horizontal range after the nth bounce.

(iv) By considering the sum of a geometric series, calculate the total horizontal distance travelled up to the sixth bounce.

⑨ A small marble is projected horizontally over the edge of a table 0.8 m high at a speed of 2.5 m s^{-1} and bounces on smooth horizontal ground with coefficient of restitution 0.7.

Calculate:

(i) the components of the velocity of the marble just before it hits the ground

(ii) its horizontal distance from the edge of the table when it first hits the ground

(iii) the horizontal distance travelled between the first and second bounces

(iv) the horizontal distance travelled between the nth and $(n + 1)$th bounces

(v) the number of bounces before the distance between bounces is less than 20 cm.

⑩ A smooth snooker ball moving at 2 m s^{-1} hits a cushion at an angle of 30° to the cushion.

The ball then rebounds and hits a second cushion which is perpendicular to the first. The coefficient of restitution for both impacts is 0.8.

(i) Find the direction of motion after each impact,

(ii) Find the magnitude of the velocity after the second impact.

(iii) Repeat parts (i) and (ii) for a ball moving at u m s^{-1} which hits the first cushion at an angle α. Assume the coefficient of restitution is e. Hence show that the direction of a ball is always reversed after hitting two perpendicular cushions and state the factor by which its speed is reduced.

⑪ Two circular discs slide on a smooth horizontal surface. Disc A has mass 6 kg and disc B has mass14 kg and both are initially at rest. A force of 12 N acts on disc A for 4 seconds and this disc then collides directly with disc B. The coefficient of restitution between the two discs is 0.25.

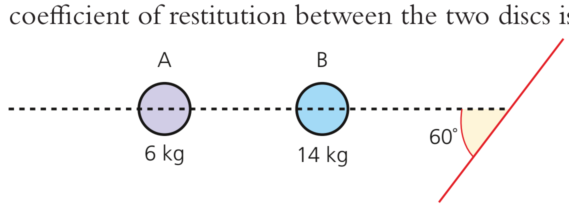


Figure 6.35

(i) Calculate the velocity of disc A before the collision.

(ii) Show that, after the collision, disc B has speed $3\,\mathrm{m\,s^{-1}}$ and find the new speed of disc A.

(iii) Calculate the impulse on disc A in this collision.

Disc B now collides with a smooth surface at 60° to the line of its motion, as shown in the diagram. The speed of the disc after the collision is $1.6\,\mathrm{m\,s^{-1}}$.

(iv) Calculate the coefficient of restitution between the disc and the plane.

⑫ A ball is projected horizontally from a table top of height h with speed u. The coefficient of restitution between the ball and the ground is e.

(i) Find an expression for the time during which the ball is moving.

(ii) Find the total horizontal distance travelled.

⑬ A ball falls vertically and strikes a fixed plane inclined at an angle θ ($\theta < 45°$) to the horizontal. The coefficient of restitution is $\frac{7}{25}$ and the ball rebounds horizontally.

(i) Show that $\tan\theta = \frac{1}{5}\sqrt{7}$.

(ii) Show that the fraction of kinetic energy lost in the collision is $\frac{18}{25}$.

5 Oblique impact of smooth elastic spheres

> ! This is section is extension material, stretching beyond the specification.

Two smooth spheres A of mass m_A and B of mass m_B collide. Immediately before impact the velocity of A is u at an angle α with the line of centres of the spheres, and the velocity of B is v at an angle β with the line of centres.

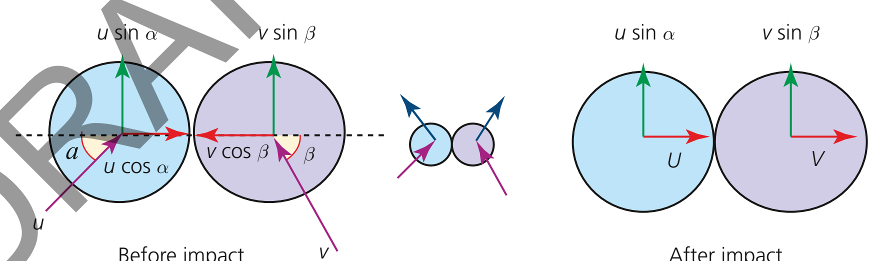


Figure 6.36

When analysing an impact like this you need to consider the components of the motion in two directions: perpendicular to the line of centres and along the line of centres.

Motion perpendicular to the line of centres

There is no impulse between the spheres in the direction of their common tangent at the point of contact and the momentum of each sphere in this direction is unchanged by the impact.

Hence the component velocities after the collision in this direction are $u \sin \alpha$ for A and $v \sin \beta$ for B.

Motion along the line of centres

Calling the components of the velocities in the direction of the line of centres U and V, conservation of momentum gives:

$$m_A u\cos\alpha - m_B v\cos\beta = m_A U + m_B V$$

The coefficient of restitution is e so that Newton's law of impact gives:

$$e(u\cos\alpha + v\cos\beta) = V - U$$

Speed of approach.

Speed of separation.

These two simultaneous equations are sufficient to determine U and V.

Once U and V are known, the velocities of A and B after the collision can be written in vector form as:

$$\mathbf{v}_A = U\mathbf{i} + u\sin\alpha\,\mathbf{j} \quad \text{and} \quad \mathbf{v}_B = V\mathbf{i} + v\sin\beta\,\mathbf{j}$$

where the directions along and perpendicular to the line of centres are denoted by **i** and **j**.

Example 6.14

In a game of snooker, the cue ball moving with speed $2\,\text{m s}^{-1}$ strikes a stationary red ball. The cue ball is moving at an angle of 60° to the line of centres of the two balls. Both balls are smooth and have the same mass m. The coefficient of restitution between the balls is $\frac{1}{2}$.

Find the velocities of the two balls after impact.

Solution

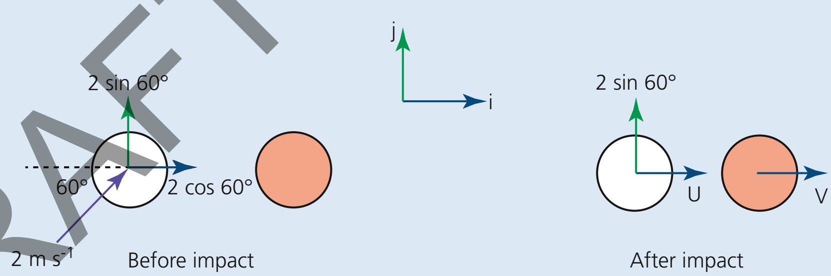


Figure 6.37

Let the component velocities in the direction of the line of centres be U for the cue ball and V for the red ball.

Conservation of momentum in the direction of the line of centres (**i**):

$$m \times 2\cos 60° = mU + mV$$

$$1 = U + V \quad \text{(A)}$$

Divide by m and use $\cos 60° = 0.5$

Newton's law of impact:

$$e \times 2\cos 60° = V - U$$

Speed of separation.

Speed of approach.

$$\frac{1}{2} = V - U \quad \text{(B)}$$

$$\text{(A)+(B)} \qquad \frac{3}{2} = 2V \Rightarrow V = \frac{3}{4}$$

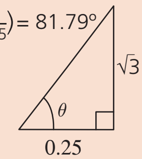
$$\text{(A)–(B)} \qquad \frac{1}{2} = 2U \Rightarrow U = \frac{1}{4}$$

The velocity of the red ball is $\frac{3}{4}\mathbf{i}$, the velocity of the white ball is $\frac{1}{4}\mathbf{i} + 2\sin 60°\,\mathbf{j} = \frac{1}{4}\mathbf{i} + \sqrt{3}\mathbf{j}$.

The red ball moves with speed $0.75\,\text{m s}^{-1}$ along the line of centres. The cue ball moves with speed $1.75\,\text{m s}^{-1}$ at 82° to the line of centres.

$\sqrt{\left(\frac{1}{4}\right)^2 + 3} = \sqrt{\frac{49}{16}} = \frac{7}{4}$

$\theta = \arctan\left(\frac{\sqrt{3}}{0.25}\right) = 81.79°$



Example 6.15

A smooth sphere A of mass $2m$, moving with speed $4\,\text{m s}^{-1}$ collides with a smooth sphere B of mass m moving with speed $2\,\text{m s}^{-1}$. The velocity of A immediately before impact makes an angle of 45° to the line of centres. The velocity of B immediately before impact is at 90° to the line of centres. The coefficient of restitution between the two balls is 0.6.

(i) Draw a diagram showing the situation before and after the collision.

(ii) Calculate the velocities of the two spheres after impact.

(iii) Calculate the loss of kinetic energy sustained by the system during the impact.

Solution

(i) The velocities of the two spheres are shown as well as their components along and perpendicular to the line of centres.

The components perpendicular to the line of centres ($4\sin 45° = 2\sqrt{2}$ for A and 2 for B) are not affected by the collision.

The components along the line of centres are taken to be V_A and V_B.

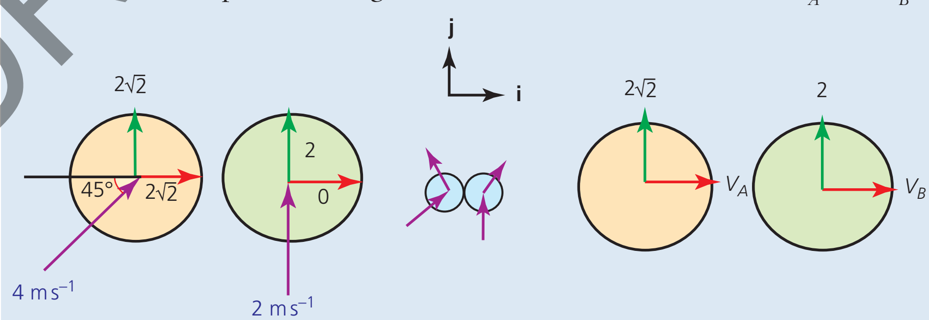


Figure 6.38

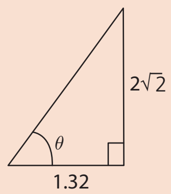
Conservation of momentum along line of centres:

(ii) $2m \times 4\cos 45° = 2mV_A + mV_B$

$4\sqrt{2} = 2V_A + V_B \qquad \text{(A)}$

Using $\cos 45° = \frac{\sqrt{2}}{2}$ and dividing by m.

$\theta = \arctan\left(\frac{2\sqrt{2}}{1.32}\right) = 65°$



Newton's law of impact:

$e \times 4\cos 45° = V_B - V_A \qquad 2\sqrt{2}e = V_B - V_A \qquad \text{(B)}$

(A)–(B): $(4-2e)\sqrt{2} = 3V_A \qquad V_A = \frac{\sqrt{2}}{3}(4-2e)$

$4 - 2e = 4 - 2 \times 0.6$

$V_A = \frac{2.8\sqrt{2}}{3} = 1.319\ldots$

(A)+2(B): $4(1+e)\sqrt{2} = 3V_B$

$V_B = \frac{4\sqrt{2}}{3}(1+e)$

$4(1+e) = 4(1+0.6)$

$V_B = \frac{6.4\sqrt{2}}{3} = 3.016\ldots$

$\sqrt{1.32^2 + (2\sqrt{2})^2} = \sqrt{1.32^2 + 8} = 3.12$

Velocity of A after impact: $1.32\mathbf{i} + 2\sqrt{2}\,\mathbf{j}$

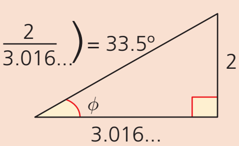
A moves with speed $3.12\,\text{m s}^{-1}$ at an angle of 65° to the line of centres.

Velocity of B after impact: $3.02\mathbf{i} + 2\mathbf{j}$

B moves with speed $3.62\,\text{m s}^{-1}$ at an angle of 33.5° to the line of centres.

$\sqrt{3.016\ldots^2 + 2^2} = 3.62$

$\phi = \arctan\left(\frac{2}{3.016\ldots}\right) = 33.5°$



(iii) The kinetic energy of the system before the collision is equal to:

$$\frac{1}{2} \times 2m \times 4^2 + \frac{1}{2} \times m \times 2^2 = 18m$$

The kinetic energy of the system after the collision is equal to:

$$\frac{1}{2} \times 2m \times 3.12^2 + \frac{1}{2} \times m \times 3.62^2 = 16.29m$$

The loss in kinetic energy is:

$$18m - 16.29m = 1.71m$$

Note

This result could have been obtained by consideration of the contribution from the components of velocity along the line of centres only as there is no change arising from the components perpendicular to the line of centres:

K.E. before: $\frac{1}{2} \times 2m \times \left(2\sqrt{2}\right)^2 + \frac{1}{2} \times m \times 0 = 8m$

K.E. after: $\frac{1}{2} \times 2m \times \frac{\left(2.8\sqrt{2}\right)^2}{3} + \frac{1}{2} \times m \times \frac{\left(6.4\sqrt{2}\right)^2}{3} = 6.293\ldots$

Loss in K.E. $8m - 6.293\ldots m = 1.71m$

Exercise 6.5

In questions 1–6, a smooth sphere A of mass m_A collides with a smooth sphere B of mass m_B, as shown in Figure 6.39.

The coefficient of restitution between the spheres is e.

Immediately before the collision, A is moving with speed u_A at an angle α with the line of centres and B is moving with speed u_B at an angle β with the line of centres.

Immediately after impact, A is moving with speed v_A at an angle α_A with the line of centres and B is moving with speed v_B at an angle β_B with the line of centres.

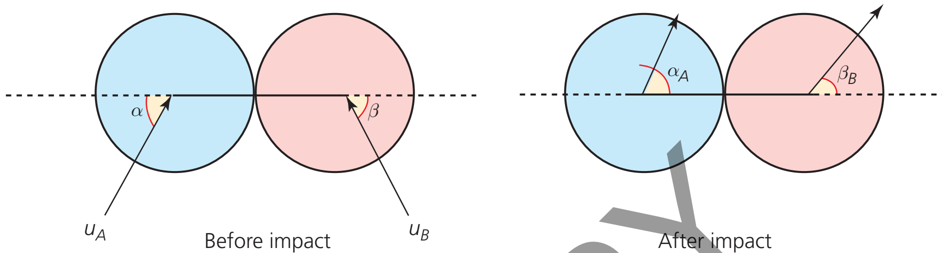


Figure 6.39

① $m_A = 4\,\text{kg}$, $m_B = 2\,\text{kg}$, $u_A = 2\,\text{m s}^{-1}$, $u_B = 4\,\text{m s}^{-1}$, $\alpha = \beta = 45°$, $e = 0.5$.
Calculate v_A, v_B, α_A and β_B.

② $m_A = m_B = m$, $\alpha = 60°$, $u_A = u$, $u_B = 0$, $e = 0.6$.
Calculate v_A, v_B, α_A and β_B.

③ $m_A = m_B = m$, $u_A = u_B = u$, $\alpha = 0°$, $\beta = 90°$, $e = 0.5$.
Calculate v_A, v_B, α_A and β_B.

④ $m_A = m_B = m$, $u_A = u_B = u$, $\alpha = 60°$, $\beta = 60°$, $e = 0.5$.
Calculate v_A, v_B, α_A and β_B.

⑤ $m_A = m$, $m_B = 5m$, $u_A = u$, $u_B = 0$, $\alpha = 60°$, $\alpha_A = 90°$.
Calculate e.

⑥ $m_A = m_B = m$, $u_B = 0$, $\alpha = 45°$, $e = \frac{2}{3}$.
Calculate α_A.

⑦ Two identical smooth balls of mass m are moving with equal speed u in opposite directions. The balls collide obliquely, so that the line of centres between the balls is at 30° to the direction of motion. Show that the loss in kinetic energy due to the impact is 75% of what it would be if the impact were direct.

⑧ A smooth sphere A of mass $2m$ moving with speed $2u$ collides with a smooth sphere B of mass m moving with speed u.

At the moment of impact, A is moving at 60° to the line of centres and B is moving at 90° to the line of centres.

The coefficient of restitution between the spheres is 0.5.

(i) Find the component of velocity along the line of centres after impact for each sphere.

(ii) Find the velocities of the spheres after impact.

(iii) Find the loss in kinetic energy for the system.

⑨ In this question all the discs are circular and have the same radius.

(i) A disc of mass m is sliding across a table when it collides with a stationary disc with the same mass. After the collision, the directions of motion of the two discs are at right angles.

Prove that the collision is perfectly elastic.

(ii) On another occasion the disc of mass m collides with a stationary disc of mass km, where $k > 1$, and the directions of their subsequent motion are at right angles. The coefficient of restitution is e.

Prove that $e = \frac{1}{k}$.

(iii) State a modelling assumption required for parts (i) and (ii).

⑩ The diagram illustrates a collision between two smooth spheres of equal mass m. Initially they are moving along parallel lines but in opposite directions. At impact the acute angle between their line of centres and the directions of their original movement is α. The coefficient of restitution in the collision is e. Before the impact both spheres have speed u.

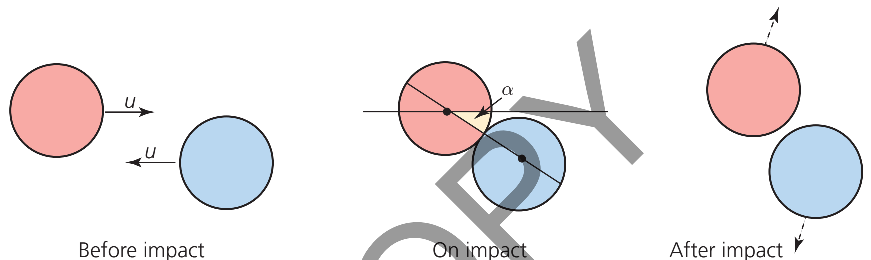


Figure 6.40

(i) Show that the loss of kinetic energy in the collision is $mu^2 \cos^2 \alpha \left(1 - e^2\right)$.

(ii) Show that, in the case when $\alpha = 30°$ and $e = \frac{1}{3}$, the direction of motion of each of the spheres after impact is at right angles to its direction before impact.

⑪ Figure 6.41 shows a white snooker ball with centre W and a red one with centre R. The straight line through W and R passes through the centre, C, of one of the pockets. WR = 100 cm and RC = 40 cm.

Apart from their colour the balls are identical. Their diameter is 5.25 cm.

A model for this snooker table is that a ball will go into the pocket if the line of approach of its centre passes within 2.5 cm of the centre of the pocket

Figure 6.41

(i) A player tries to hit the white ball exactly along the straight line WRC so that the red ball will go into the pocket. Instead, however, he hits the ball at an angle α to the required line.

What happens to the red ball in the cases when:

(a) $\alpha = 0.5°$

(b) $\alpha = 0.15°$.

Figure 6.42 shows a different situation. There are two red balls with centres R_1 and R_2 on the line WC. R_1R_2 = 20 cm and R_2C = 20 cm.

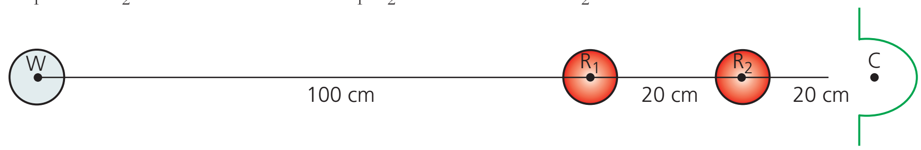


Figure 6.42

The player hits the white ball into the first red ball, which then hits the second red ball. If the shot is successful the second red ball goes into the pocket. A shot like this is called a 'plant'.

The player hits the white ball at an angle α to the line WC.

(ii) Show that, if $\alpha = 0.15°$, the shot is not successful.

(iii) Find, to 2 significant figures, the smallest value of α for which the shot is unsuccessful.

(iv) Explain why the plant is not a popular shot with serious snooker players if the balls are not close to each other.

KEY POINTS

1. The impulse from a force **F** is given by **F***t* where *t* is the time for which the force acts.
2. Impulse is conventionally denoted by **J**. It is a vector quantity.
3. The momentum of a body of mass *m* travelling with velocity **v** is given by *m***v.** Momentum is a vector quantity.
4. The S.I. unit of impulse and momentum is the newton second (N s)
5. The impulse–momentum equation is

$$\text{impulse} = \text{final momentum} - \text{initial momentum}$$

6. The law of conservation of momentum states that when no external forces are acting on a system, the total momentum of the system is constant. Since momentum is a vector quantity this applies to the magnitude of the momentum in any direction.
7. Newton's law of impact:

Coefficient of restitution $e = \dfrac{\text{speed of separation}}{\text{speed of approach}}$

$$\text{speed of separation} = e \times \text{speed of approach}$$

8. Collision between a sphere and a fixed plane

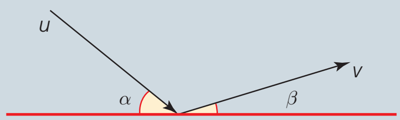


Figure 6.43

Component of velocity parallel to surface remains unchanged [$v \cos\beta = u \cos\alpha$]
Component of velocity perpendicular to surface: [$v \sin\beta = -eu \sin\alpha$]

Loss in kinetic energy: $\frac{1}{2}mu^2\sin^2\alpha\left(1 - e^2\right)$

9 Oblique impact between smooth spheres

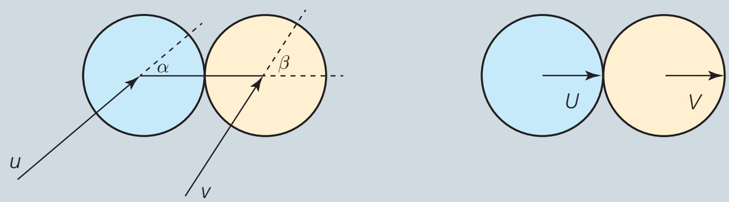


Figure 6.44

Perpendicular to line of centres

$u \sin \alpha$ and $v \sin \beta$ remain unchanged by the collision.

Along line of centres

Conservation of momentum:

$$m_A u \cos\alpha + m_B v \cos\beta = m_A U + m_B V \quad ①$$

Newton's law of impact:

$$V - U = e(u\cos\alpha - v\cos\beta) \quad ②$$

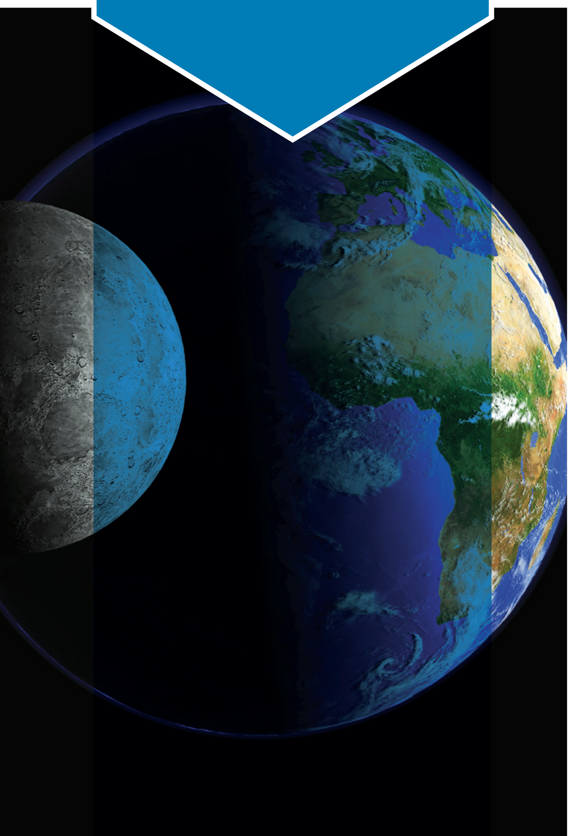
Equations ① and ② can be solved to find U and V.

LEARNING OUTCOMES

When you have completed this chapter, you should:

- know how to apply the principle of conservation of momentum to direct impacts
- understand Newton's law of impact and know the meaning of coefficient of restitution
- know and use the fact that $0 \le e \le 1$
- understand the implications of values of 0 and 1 for the coefficient of restitution
- understand that when the coefficient of restitution is less than 1, energy is not conserved during an impact
- be able to find the loss of kinetic energy during a direct impact
- know that for perfectly elastic collisions there is no energy loss
- know that for perfectly inelastic collisions the energy loss is the largest it can be
- understand the term oblique impact and the assumptions made when modelling oblique impact
- know the meaning of Newton's experimental law and of the coefficient of restitution when applied to an oblique impact
- be able to solve problems involving impact between an object and a fixed smooth plane by considering components of motion parallel and perpendicular to the line of impulse
- be able to solve problems involving impact between two spheres by considering components of motion in directions parallel and perpendicular to the line of centres
- be able to calculate the loss of kinetic energy in an oblique impact
- be able to tackle impulse problems where the force varies with time
- be able to tackle impulse problems in more than one dimension.

7 Circular motion



Whirlpools and storms
his circling arm invest;
With all the might of
gravitation blest.
Alexander Pope

- These photographs show some objects that move in circular paths. What other examples can you think of?
- What makes objects move in circles?
- Why does the moon circle the earth?
- What happens to the hammer when the athlete lets it go?
- Do the pilots of the planes need to be strapped into their seats at the top of a loop in order not to fall out?

1 Introduction to circular motion

The answers to the questions on the previous page lie in the nature of circular motion. Even if an object is moving at constant speed in a circle, its velocity keeps changing because its direction of motion keeps changing. Consequently the object is accelerating and so, according to Newton's first law, there must be a force acting on it. The force required to keep an object moving in a circle can be provided in many ways.

Without the Earth's gravitational force, the Moon would move off at constant speed in a straight line into space. The wire attached to the athlete's hammer provides a tension force which keeps the ball moving in a circle. When the athlete lets go, the ball flies off at a tangent because the tension has disappeared.

Although it would be sensible for the pilot to be strapped in, no upward force is necessary to stop him falling out of the plane because his weight contributes to the force required for motion in a circle.

In this chapter, these effects are explained.

Units

For the early study of mathematics, angles are measured in degrees, where 360° is a whole turn. In more advanced mathematics, angles are measured in a different unit, called the radian, where 2π radians is a whole turn. An introduction to the radian can be found at the back of this book as an appendix.

Notation

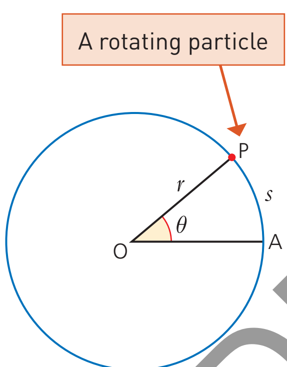


Figure 7.1

To describe circular motion (or indeed any other topic) mathematically you need a suitable notation. It will be helpful in this chapter to use the notation (attributed to Newton) for differentiation with respect to time in which, for example, $\frac{\mathrm{d}s}{\mathrm{d}t}$ is written as $\dot{s}$, and $\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$ as $\ddot{\theta}$.

Figure 7.1 shows a particle P moving round the circumference of a circle of radius r, centre O. At time t, the position vector OP of the particle makes an angle θ (in radians) with the fixed direction OA. The arc length AP is denoted by s.

Angular speed

Using this notation,

$$s = r\theta$$

Differentiating with respect to time using the product rule gives

$$\frac{\mathrm{d}s}{\mathrm{d}t} = r\frac{\mathrm{d}\theta}{\mathrm{d}t} + \theta\frac{\mathrm{d}r}{\mathrm{d}t}.$$

Since r is constant for a circle, $\frac{\mathrm{d}r}{\mathrm{d}t} = 0$, so the rate at which the arc length increases is

$$\frac{\mathrm{d}s}{\mathrm{d}t} = r\frac{\mathrm{d}\theta}{\mathrm{d}t} \text{ or } \dot{s} = r\dot{\theta}.$$

In this equation, $\dot{s}$ is the speed at which P is moving round the circle (often denoted by v), and $\dot{\theta}$ is the rate at which the angle θ is increasing, i.e. the rate at which the position vector $\overrightarrow{\mathrm{OP}}$ is rotating.

The quantity $\frac{\mathrm{d}\theta}{\mathrm{d}t}$, or $\dot{\theta}$, can be called the *angular velocity* or the *angular speed* of P.

In more advanced work, angular velocity is treated as a vector, whose direction is taken to be that of the axis of rotation. In this book, $\frac{d\theta}{dt}$ is often referred to as angular speed, but is given a sign: positive for an anticlockwise rotation and negative for a clockwise rotation.

> **Note**
> It is common practice to give angular speed as multiples of π

Angular speed is often denoted by ω, the Greek letter omega. So the equation $\dot{s} = r\dot{\theta}$ may be written as

$$v = r\omega$$

Notice that for this equation to hold, θ must be measured in radians, so the angular speed is measured in radians per second or $\text{rad}\,\text{s}^{-1}$.

Figure 7.2 shows a disc rotating about its centre, O, with angular speed ω. The line OP represents any radius.

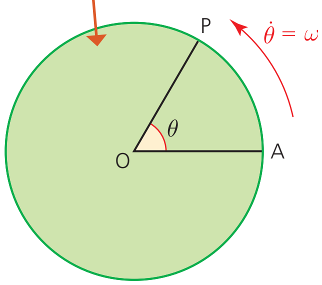


Figure 7.2

Every point on the disc describes a circular path, and all the points have the same angular speed. However, the actual speed of any point depends on its distance from the centre: increasing r in the equation $v = r\omega$ increases v. You will appreciate this if you have ever been at the end of a rotating line of people in a dance or watched a body of marching soldiers wheeling round a corner.

Angular speeds are sometimes measured in revolutions per second or revolutions per minute (rpm) where one revolution is equal to 2π radians. For example, a computer hard disc might spin at 7200 rpm or more; at cruising speeds, crankshafts in car engines typically rotate at 3000 to 4000 rpm.

Example 7.1

A police car drives at 40 mph around a circular bend of radius 16 m. A second car moves so that it has the same angular speed as the police car but in a circle of radius 12 m. Is the second car breaking the 30 mph speed limit? (Use the approximation 1 mile = $\frac{8}{5}$ km.)

Solution

Converting miles per hour to metres per second gives:

$$40\text{ mph} = 40 \times \frac{8}{5}\,\text{km}\,\text{h}^{-1}$$
$$= \frac{40 \times 8 \times 1000}{5 \times 3600}\,\text{m}\,\text{s}^{-1}$$
$$= \frac{160}{9}\,\text{m}\,\text{s}^{-1}$$

Using $v = r\omega$

$$\omega = \frac{160}{9 \times 16}\,\text{rad}\,\text{s}^{-1}$$
$$= \frac{10}{9}\,\text{rad}\,\text{s}^{-1}$$

The speed of the second car is:

$$v = 12\omega$$
$$= \frac{10}{9} \times 12\,\text{m}\,\text{s}^{-1}$$
$$= \frac{120 \times 5 \times 3600}{9 \times 8 \times 1000}\,\text{mph}$$
$$= 30\,\text{mph}$$

The second car is just on the speed limit.

> **Notes**
> 1 Notice that working in fractions gives an exact answer.
> 2 A quicker way to do this question would be to notice that, because the cars have the same angular speed, the actual speeds of the cars are proportional to the radii of the circles in which they are moving. Using this method it is possible to stay in mph. The ratio of the two radii is $\frac{12}{16}$ so the speed of the second car is $\frac{12}{16} \times 40$ mph = 30 mph.

Exercise 7.1

① Find the angular speed in radians per second to one decimal place, of antique records rotating at:

(i) 78 rpm (ii) 45 rpm (iii) $33\frac{1}{3}$ rpm.

② A flywheel is rotating at $300\,\text{rad}\,\text{s}^{-1}$. Express this angular speed in rpm, correct to the nearest whole number.

③ The London Eye observation wheel has a diameter of 135 m and completes one revolution in 30 minutes.

(i) Calculate its angular speed in:

(a) rpm (b) radians per second.

(ii) Calculate the speed of the point on the circumference where passengers board the moving wheel.

④ A lawnmower engine is started by pulling a rope that has been wound round a cylinder of radius 4 cm. Find the angular speed of the cylinder at a moment when the rope is being pulled with a speed of $1.3\,\text{m}\,\text{s}^{-1}$. Give your answer in radians per second, correct to one decimal place.

⑤ The wheels of a car have radius 20 cm. What is the angular speed, in radians per second, correct to one decimal place, of a wheel when the car is travelling at:

(i) $10\,\text{m}\,\text{s}^{-1}$ (ii) $30\,\text{m}\,\text{s}^{-1}$?

⑥ The angular speed of an audio CD changes continuously so that a laser can read the data at a constant speed of $12\,\text{m}\,\text{s}^{-1}$. Find the angular speed (in rpm) when the distance of the laser from the centre is:

(i) 30 mm (ii) 55 mm.

⑦ What is the average angular speed of the Earth in radians per second as it

(i) orbits the Sun

(ii) rotates about its own axis?

The radius of the Earth is 6400 km.

(iii) At what speed is someone on the equator travelling relative to the centre of the Earth?

(iv) At what speed are you travelling relative to the centre of the Earth?

⑧ A tractor has front wheels of diameter 70 cm and back wheels of diameter 1.6 m.

What is the ratio of their angular speeds when the tractor is being driven along a straight road?

⑨ (i) Find the kinetic energy of a 50 kg person riding a big wheel with radius 5 m when the ride is rotating at 3 rpm. You should assume that the person can be modelled as a particle.

(ii) Explain why this modelling assumption is necessary.

⑩ The minute hand of a clock is 1.2 m long and the hour hand is 0.8 m long.

(i) Find the speeds of the tips of the hands.

(ii) Find the ratio of the speeds of the tips of the hands and explain why this is not the same as the ratio of the angular speeds of the hands.

⑪ Figure 7.3 represents a ‘Chairoplane’ ride at a fair. It completes one revolution every 2.5 seconds.

(i) Find the radius of the circular path which a rider follows.

(ii) Find the speed of a rider.

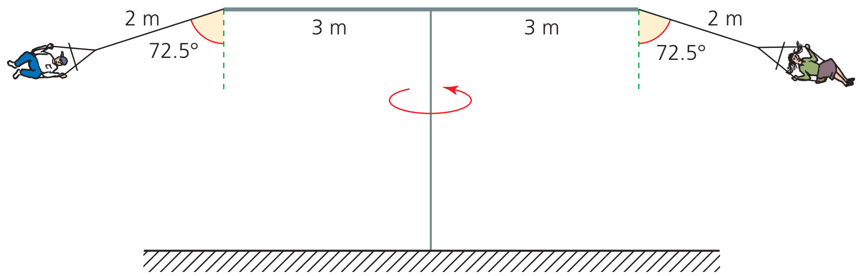


Figure 7.3

Velocity and acceleration

Velocity and acceleration are both vector quantities. They can be expressed either in magnitude–direction form, or in components. When describing circular motion or other orbits, it is most convenient to take components in directions along the radius (*radial* direction) and at right angles to it (*transverse* direction).

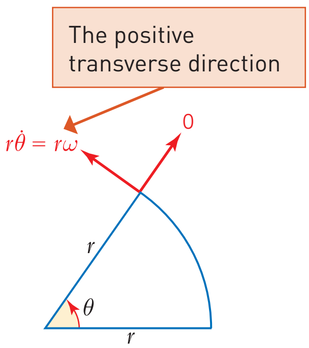


Figure 7.4 Velocity

For a particle moving round a circle of radius r, the velocity has:

radial component: 0

transverse component: $r\dot{\theta}$ or $r\omega$.

The acceleration of a particle moving round a circle of radius r has:

radial component: $-r\dot{\theta}^2$ or $-r\omega^2$

transverse component: $r\ddot{\theta}$ or $r\dot{\omega}$.

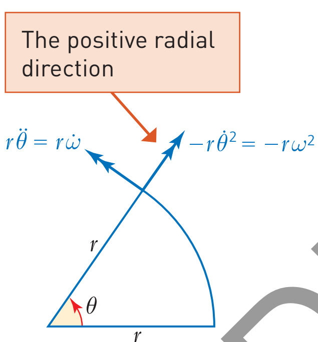


Figure 7.5 Acceleration

The transverse component is just what you would expect, the radius multiplied by the angular acceleration, $\ddot{\theta}$. If the particle has constant angular speed, its angular acceleration is zero and so the transverse component of its acceleration is also zero.

In contrast, the radial component of the acceleration, $-r\omega^2$, is almost certainly not a result you would have expected intuitively. It tells you that a particle travelling in a circle is always accelerating towards the centre of the circle, but without getting any closer to the centre. If this seems a strange idea, you may find it helpful to remember that circular motion is not a natural state; left to itself a particle will travel in a straight line. To keep a particle in the unatural state of circular motion it must be given an acceleration at right angles to its motion, i.e. towards the centre of the circle.

The derivation of these expressions for the acceleration of a particle in a circular motion is complicated by the fact that the radial and transverse directions are themselves changing as the particle moves round the circle, in contrast to the fixed x and y directions in the Cartesian system. The derivation is given in a mathematical note on page 316. At first reading you may prefer to accept the results, but make sure that at a later stage you work through and understand the derivation.

2 Circular motion with constant speed

In this section, the circular motion is assumed to be uniform and so have no transverse component of acceleration. Later in the chapter, situations are considered in which the angular speed varies.

Problems involving circular motion often refer to the actual speed of the object, rather than its angular speed. It is easy to convert one into the other using the relationship $v = r\omega$. The relationship can also be used to express the magnitude of the acceleration in terms of v and r.

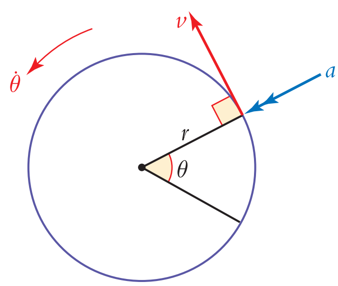


Figure 7.6

$$\omega = \frac{v}{r}$$

$$a = r\omega^2 = r\left(\frac{v}{r}\right)^2$$

$$\Rightarrow \quad a = \frac{v^2}{r} \text{ towards the centre}$$

Velocity $v = r\omega$

Angular speed $\dot{\theta} = \omega$

Acceleration has magnitude $a = r\omega^2 = \frac{v^2}{r}$ and is directed towards the centre

Example 7.2

A turntable is rotating at 45 rpm. A fly is standing on it, 8 cm from its centre.

Find

(i) the angular speed of the fly in radians per second

(ii) the speed of the fly in metres per second

(iii) the acceleration of the fly.

Solution

(i) $45 \text{ rpm} = 45 \times 2\pi \text{ rad min}^{-1}$

$$= \frac{45 \times 2\pi}{60} \text{ rad s}^{-1}$$

$$= \frac{3\pi}{2} \text{ rad s}^{-1}.$$

(ii) v can be found using

$$v = r\omega$$

$$= 0.08 \times \frac{3\pi}{2}$$

$$= 0.376...$$

So the speed of the fly is 0.38 m s^{-1} (to 2 dp).

(iii) The acceleration of the fly is given by

$$r\omega^2 = 0.08 \times \left(\frac{3\pi}{2}\right)^2 \text{ m s}^{-2}$$

$$= 1.78 \text{ m s}^{-2}$$

It is directed towards the centre of the record.

Hint

One revolution is 2π radians.

The forces required for circular motion

Newton's first law of motion states that a body will continue in a state of rest or uniform motion in a straight line unless acted upon by an external force. Any object moving in a circle, such as the police car and the fly in Examples 6.1 and 6.2, must therefore be acted upon by a resultant force in order to produce the required acceleration towards the centre.

A force towards the centre is called a *centripetal* (centre-seeking) force. A resultant centripetal force is necessary for a particle to move in a circular path.

Examples of circular motion

You are now in a position to use Newton's second law to determine theoretical answers to some of the questions which were posed at the beginning of this chapter. These will, as usual, be obtained using models of the true motion which will be based on simplifying assumptions, for example zero air resistance.

Example 7.3

A coin is placed on a rotating horizontal turntable, 5 cm from the centre of rotation. The coefficient of friction between the coin and the turntable is 0.5.

(i) The speed of rotation of the turntable is gradually increased. At what angular speed will the coin begin to slide?

(ii) What happens next?

Solution

(i) Because the speed of the turntable is increased only gradually, the coin will not slip tangentially.

Figure 7.7 shows the forces acting on the coin, and its acceleration.

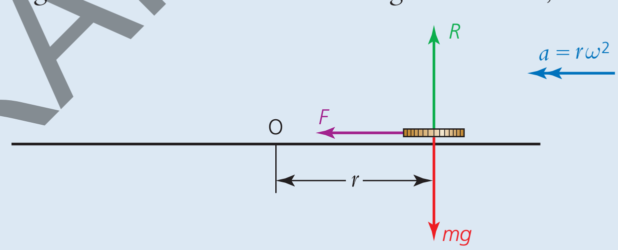


Figure 7.7

The acceleration is towards the centre of the circular path, O, so there must be a frictional force F in that direction.

There is no vertical component of acceleration, so the resultant force acting on the coin has no vertical component.

Therefore

$$R - mg = 0$$

$$R = mg \quad ①$$

By Newton's second law towards the centre of the circle:

$$\text{Force } F = ma = mr\omega^2$$

The coin will not slide so long as $F \le \mu R$. ②

Substituting from ① and ② this gives

$$mr\omega^2 \leq \mu mg$$

$$\Rightarrow \quad r\omega^2 \leq \mu g$$

Notice that the mass, m, has been eliminated at this stage, so the answer does not depend upon it.

Taking g in $\mathrm{m\,s^{-2}}$ as 9.8 and substituting $r = 0.05$ and $\mu = 0.5$

$$\omega^2 \leq 98$$

$$\omega \leq \sqrt{98}$$

$$\omega \leq 9.89\ldots$$

The coin will move in a circle provided the angular speed is less than about $10\,\mathrm{rad\,s^{-1}}$ and the speed is independent of the mass of the coin.

(ii) When the angular speed increases beyond this, the coin slips off the turntable. When it reaches the edge, it will fly off in the direction of the tangent.

3 The conical pendulum

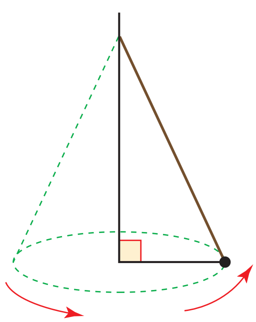


Figure 7.8

A conical pendulum consists of a small bob tied to one end of a string. The other end of the string is fixed and the bob is made to rotate in a horizontal circle below the fixed point so that the string describes a cone, as in Figure 7.8.

EXPERIMENT

1 Draw a diagram showing the magnitude and direction of the acceleration of the bob and the forces acting on it.

2 In the case that the radius of the circle remains constant, try to predict the effect of the angular speed when the length of the string is increased or when the mass of the bob is increased. What might happen when the angular speed increases?

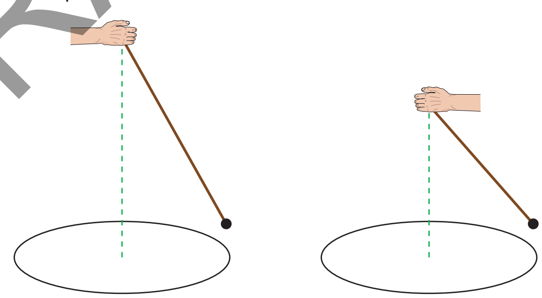


Figure 7.9

3 Draw two circles of equal diameter on horizontal surfaces so that two people can make the bobs of two conical pendulums rotate in circles of the same radius.

(i) Compare pendulums of different lengths with bobs of equal mass.

(ii) Compare pendulums of the same length but with bobs of different masses.

Does the angular speed depend on the length of the pendulum or the mass of the bob?

4 What happens when somebody makes the speed of the bob increase?

5 Can the bob be made to rotate with the string horizontal?

When you are answering questions on the conical pendulum, you need to look carefully at the level of accuracy to which g is given. Your final answers should always match the accuracy of all the given information.

Theoretical model for the conical pendulum

A conical pendulum may be modelled as a particle of mass m attached to a light, inextensible string of length l. The mass is rotating in a horizontal circle with angular speed ω and the string makes an angle α with the downward vertical. The radius of the circle is r and the tension in the string is T, all in consistent units (e.g. S.I. units). The situation is shown in Figure 7.10.

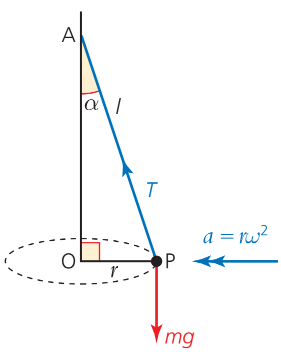


Figure 7.10

The magnitude of the acceleration is $r\omega^2$. The acceleration acts in a horizontal direction towards the centre of the circle. This means that there must be a resultant force acting towards the centre of the circle.

There are two forces acting on this particle, its weight mg and the tension T in the string.

As the acceleration of the particle has no vertical component, the resultant force has no vertical component, so

$$T\cos\alpha - mg = 0 \qquad ①$$

Using Newton's second law towards the centre, O, of the circle

$$T\sin\alpha = ma = mr\omega^2 \qquad ②$$

In triangle AOP $\quad r = l\sin\alpha$

Substituting for r in ② gives $\quad T\sin\alpha = m(l\sin\alpha)\,\omega^2$

$$\Rightarrow \quad T = ml\omega^2$$

Substituting this in ① gives

$$ml\omega^2\cos\alpha - mg = 0$$

$$\Rightarrow \quad l\cos\alpha = \frac{g}{\omega^2} \qquad ③$$

This equation provides sufficient information to give theoretical answers to the questions in the experiment.

- When r is kept constant and the length of the string is increased, the length $AO = l\cos\alpha$ increases. Equation ③ indicates that the value of $\frac{g}{\omega^2}$ increases and so the angular speed ω decreases. Conversely, the angular speed increases when the string is shortened.
- The mass of the particle does not appear in equation ③, so it has no effect on the angular speed, ω.
- When the length of the pendulum is unchanged, but the angular speed is increased, $\cos\alpha$ decreases, leading to an increase in the angle α and hence in r.
- If $\alpha \geq 90°$, $\cos\alpha \leq 0$, so $\frac{g}{\omega^2} \leq 0$, which is impossible. You can see from Figure 7.10 that the tension in the string must have a vertical component to balance the weight of the particle.

Example 7.4

An inextensible light string of length 50 cm is attached to a fixed point A that is 30 cm above a smooth table. The other end is attached to a particle of mass 3 kg at point B. The particle is moving in a circle supported by the table with the string taut.

(i) Find the value of the normal force exerted by the table on the particle when it moving with speed $10\,\text{cm}^{-3}$.

(ii) Find the speed with which the particle is moving when it is about to lift from the table

Solution

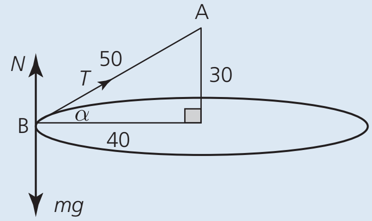


Figure 7.11

(i) The acceleration towards the centre of the circle is $\frac{v^2}{r} = \frac{100}{40} = 2.5$.

Using $F = ma$ horizontally, $T\cos\alpha = m\frac{v^2}{r}$

$$T \times \frac{4}{5} = m \times 2.5 \text{ so, } T = \frac{25}{8}m$$

Notice the use of the × symbol and the fact that in the first equation the order of the terms matches that in the line above.

Resolving vertically for the mass, $N + T\sin\alpha$

$$N + \frac{3}{5} \times \frac{25}{8}m = mg$$

So if $m = 3$ and $g = 9.80$, then $N = 23.8$ N.

(ii) The acceleration towards the centre of the circle is $\frac{v^2}{r} = \frac{v^2}{40}$.

Using $F = ma$ horizontally, you find $T\cos\alpha = m\frac{v^2}{r}$

$$T \times \frac{4}{5} = \frac{v^2}{40}m \text{ so, } T = \frac{v^2}{32}m$$

Resolving vertically for the mass, $N + T\sin\alpha$,

$$N + \frac{3}{5} \times \frac{v^2}{32}m = mg$$

If the mass is lifting from the table, $N = 0$, and so $v = 22.9\,\text{cm}\,\text{s}^{-1}$

Example 7.5

The diagram on the right represents one of several arms of a fairground ride shown on the left. The arms rotate about an axis and riders sit in chairs linked to the arms by chains.

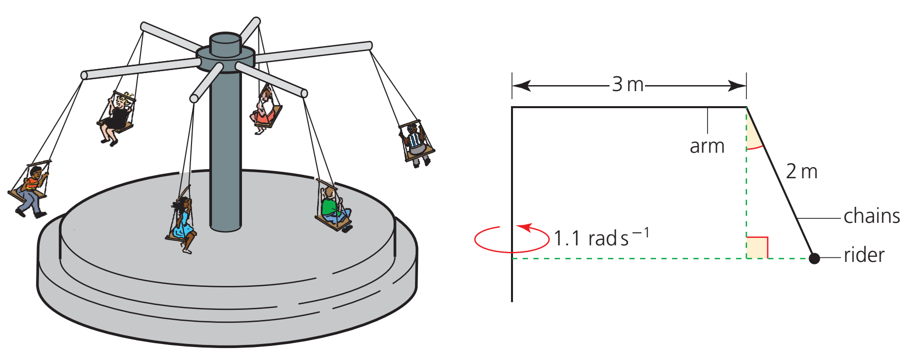


Figure 7.12

The chains are 2 m long and the arms are 3 m long. Find the angle that the chains make with the vertical when the rider rotates at $1.1\,\text{rad}\,\text{s}^{-1}$.

Solution

Let T be the resultant tension in the chains holding a chair and let m kg be the mass of the chair and rider.

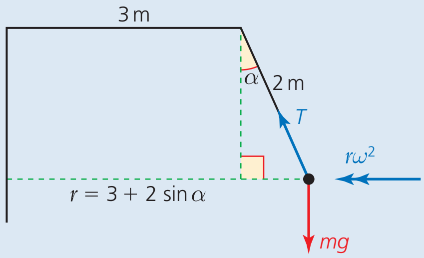


Figure 7.13

If the chains make an angle α with the vertical, the motion is in a horizontal circle with radius given by

$$r = 3 + 2\sin\alpha.$$

The magnitude of the acceleration is given by

$$r\omega^2 = (3 + 2\sin\alpha)\,1.1^2$$

It is in a horizontal direction towards the centre of the circle. Using Newton's second law in this direction gives

$$\text{Force} = mr\omega^2$$

$$\Rightarrow \quad T\sin\alpha = m(3 + 2\sin\alpha)1.1^2 \qquad ①$$

$$= 1.21\,m(3 + 2\sin\alpha)$$

Vertically $\quad T\cos\alpha - mg = 0$

$$\Rightarrow \quad T = \frac{mg}{\cos\alpha}$$

Substituting for T in equation ①:

$$\frac{mg}{\cos\alpha}\sin\alpha = 1.21\,m(3 + 2\sin\alpha)$$

$$\Rightarrow \quad 9.8\tan\alpha = 3.63 + 2.42\sin\alpha$$

Since m cancels out at this stage, the angle does not depend on the mass of the rider.

This equation cannot be solved directly, but a numerical method will give you the solution 25.5° correct to three significant figures. You might like to solve the equation yourself or check that this solution does in fact satisfy the equation.

Note

Since the answer does not depend on the mass of the rider and chair, when riders of different masses, or even no riders, are on the equipment all the chains should make the same angle with the vertical.

Sometimes a conical pendulum may feature two strings, as in the example below.

Example 7.6

A particle C of mass 0.925 kg is moving in a horizontal circle of radius 0.240 m. The centre of the circle is O and C moves with a constant angular speed of exactly 5 radians per second.

The particle is attached to points A and B by two light inextensible strings which are taut.

As shown in the diagram the points O, A and B lie on the axis of rotation. OB = 0.180 m and AB = 0.270 m.

Taking the value of g to be 9.8 m s^{-2}, find the tensions in the two strings.

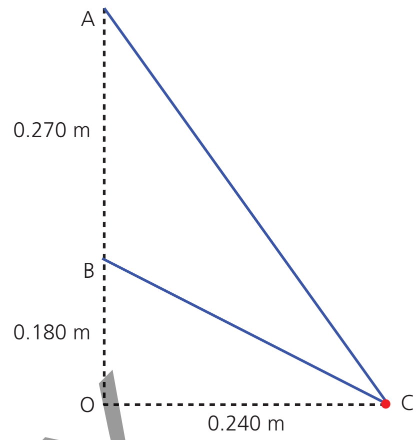


Figure 7.14

Solution

Let the tensions in the strings AC and BC be T_1 and T_2, and the angles they make with the horizontal be α and β respectively.

Using Pythagoras, $BC = \sqrt{0.24^2 + 0.18^2} = 0.3\,\text{m}$

$AC = \sqrt{0.24^2 + 0.45^2} = 0.51\,\text{m}$

So $\sin\alpha = \dfrac{0.45}{0.51} = 0.882\ldots,\quad \cos\alpha = \dfrac{0.24}{0.51} = 0.470\ldots$

and $\sin\beta = \dfrac{0.18}{0.3} = 0.6,\quad \cos\beta = \dfrac{0.24}{0.3} = 0.8$

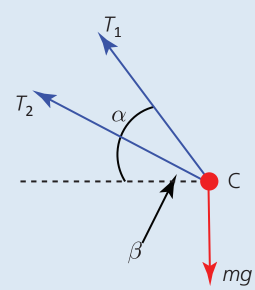


Figure 7.15

In the vertical direction the particle is in equilibrium

so $T_1 \sin\alpha + T_2 \sin\beta = mg$

$0.882\ldots \times T_1 + 0.6T_2 = 9.065$ ①

In the horizontal direction, the particle is accelerating towards the centre of the circle

so $T_1 \cos\alpha + T_2 \cos\beta = m\omega^2 r$

$0.470\ldots \times T_1 + 0.8T_2 = 0.925 \times 5^2 \times 0.24 = 5.55$ ②

Solving the simultaneous equations ① and ② gives

$T_1 = 9.258\ \ldots$ and $T_2 = 1.488\ \ldots$

So to 2 significant figures the tensions are 9.3 N in AC and 1.5 N in BC.

> Notice that the final answers have been given to only 2 significant figures even though most of the information in the question is accurate to at least 3 figures. The reason is that the value of g is stated to just 2 significant figures. If g had been given to 3 significant figures, for example as 9.80 or 9.81, the answers would also have been given to 3 figures.

> This section is extension, stretching beyond the specification.

Note

Keep away from other people and breakable objects when carrying out this activity.

4 Banked tracks

ACTIVITY 7.1

Place a coin on a piece of stiff A4 card and hold it horizontally at arm's length with the coin near your hand.

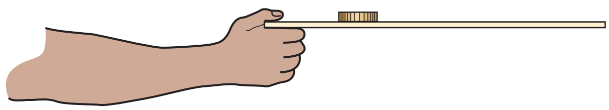


Figure 7.16

Turn round slowly so that your hand moves in a horizontal circle. Now gradually speed up. The outcome will probably not surprise you.
What happens though if you tilt the card?

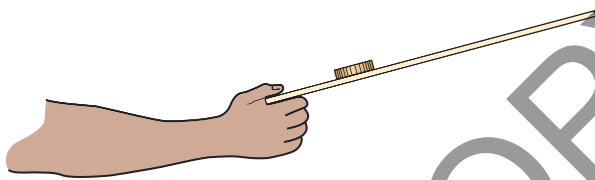


Figure 7.17

Figure 7.18

You may have noticed that when they curve round bends, most roads are banked so that the edge at the outside of the bend is slightly higher than that at the inside. For the same reason, the outer rail of a railway track is slightly higher than the inner rail when it goes round the bend. On bobsleigh tracks the bends are almost bowl shaped, with a much greater gradient on the outside.

Figure 7.19 shows a car rounding a bend on a road which is banked so that the cross-section makes an angle α with the horizontal.

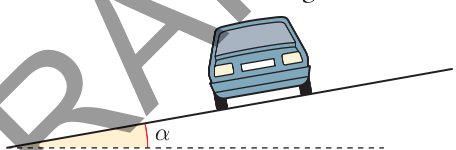


Figure 7.19

In modelling such situations, it is usual to treat the bend as part of a horizontal circle whose radius is large compared to the width of the car. In this case, the radius of the circle is taken to be r metres, and the speed of the car constant at v metres per second.

Discussion point

The direction of the frictional force F will be up or down the slope depending on whether the car has a tendency to slip sideways towards the inside or outside of the bend.

Under what conditions do you think each of these will occur?

The car is modelled as a particle which has an acceleration of $\frac{v^2}{r}\,\mathrm{m\,s^{-2}}$ in a horizontal direction towards the centre of the circle. The forces and acceleration are shown in Figure 7.20

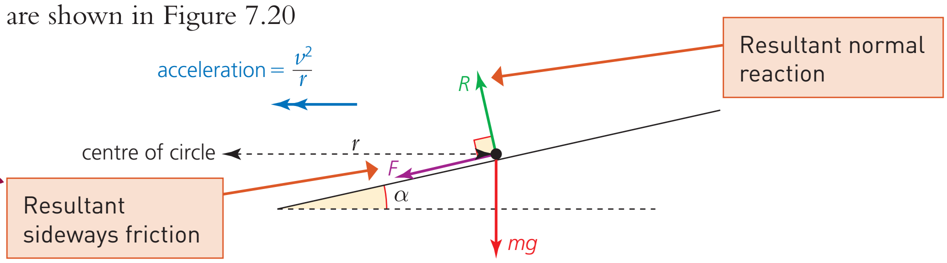


Figure 7.20

Example 7.7

A car is rounding a bend of radius 100 m which is banked at an angle of 10° to the horizontal. At what speed must the car travel to ensure that it has no tendency to slip sideways?

Solution

When there is no tendency to slip there is no frictional force, so, in the plane perpendicular to the direction of motion of the car, the forces and acceleration are as shown in Figure 7.21. The only horizontal force is provided by the horizontal component of the normal reaction of the road on the car.

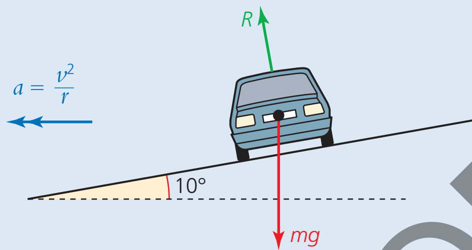


Figure 7.21

Note

The normal reaction R is resolved into components

$R \sin 10°$ horizontally ←

$R \cos 10°$ vertically ↑

Vertically there is no acceleration so there is no resultant force

$$R\cos 10° - mg = 0$$

$$\Rightarrow \quad R = \frac{mg}{\cos 10°} \qquad ①$$

By Newton's second law in the horizontal direction towards the centre of the circle

$$R\sin 10° = ma = \frac{mv^2}{r}$$

$$= \frac{mv^2}{100}$$

Substituting for R from ①

$$\left(\frac{mg}{\cos 10°}\right)\sin 10° = \frac{mv^2}{100}$$

$$\Rightarrow \quad v^2 = 100g\tan 10°$$

$$\Rightarrow \quad v = 13.14\ldots$$

The mass, m, cancels out at this stage, so the answer does not depend on it.

The speed of the car must be about 13.1 m s^{-1} or 30 mph.

There are two important points to notice in this example.

- The speed is the same whatever the mass of the car.
- The example looks at the situation when the car does not tend to slide, and finds the speed at which this is the case. At this speed, the car does not depend on friction to keep it from sliding and, indeed, it could travel safely

round the bend at this speed even in very icy conditions. However, at other speeds there is a tendency to slide, and friction actually helps the car to follow its intended path.

Safe speeds on a bend

What would happen in the previous example if the car travelled either more slowly than $13.1\,\mathrm{m\,s^{-1}}$ or more quickly?

The answer is that there would be a frictional force acting so as to prevent the car from sliding across the road.

There are two possible directions for the frictional force. When the vehicle is stationary or travelling slowly, there is a tendency to slide down the slope and the friction acts up the slope to prevent this. When it is travelling quickly round the bend, the car is more likely to slide up the slope, so the friction acts down the slope.

Fortunately, under most road conditions, the coefficient of friction between tyres and the road is large, typically about 0.8. This means that there is a range of speeds that are safe for negotiating any particular bend.

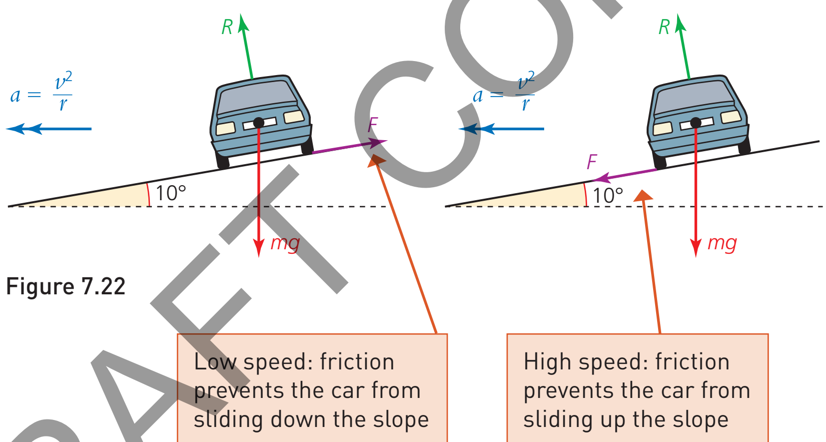


Figure 7.22

Example 7.8

A bend on a railway track has a radius of 500 m and is to be banked so that a train can negotiate it at 60 mph without the need for a lateral force between its wheels and the rail. The distance between the rails is 1.43 m.

How much higher should the outside rail be than the inside one?

Solution

There is very little friction between the track and the wheels of a train. Any sideways force required is provided by the 'lateral thrust' between the wheels and the rail. The ideal speed for the bend is such that the lateral thrust is zero.

Figure 7.21 shows the forces acting on the train and its acceleration when the track is banked at an angle α to the horizontal.

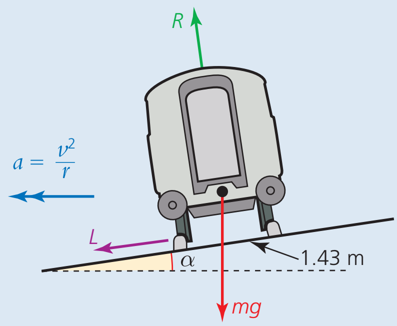


Figure 7.23

When there is no lateral thrust, $L = 0$.

Horizontally: $$R\sin\alpha = \frac{mv^2}{r} \quad ①$$

Vertically: $$R\cos\alpha = mg \quad ②$$

Dividing ① by ② gives $$\tan\alpha = \frac{v^2}{rg}$$

Using the fact that 60 mph = 26.8 m s^{-1} this becomes

$$\tan\alpha = 0.147$$

$$\Rightarrow \quad \alpha = 8.4° \text{ (to 2 s.f.)}$$

The outside rail should be raised by 1.43 sin α metres, i.e. about 21 cm.

Exercise 7.2

① Figure 7.24 shows two cars A and B, travelling at constant speeds in different lanes (radii 24 m and 20 m) round a circular traffic island. Car A has speed 18 m s^{-1} and car B has speed 15 m s^{-1}.

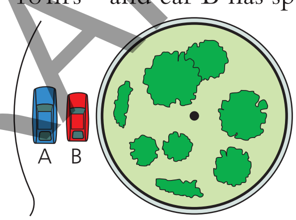


Figure 7.24

Answer the following questions, giving reasons for your answers.

(i) Which car has the greater angular speed?

(ii) Is one car overtaking the other?

(iii) Find the magnitude of the acceleration of each car.

(iv) In which direction is the resultant force on each car acting?

② Two coins are placed on a horizontal turntable. Coin A has mass 15 g and is placed 5 cm from the centre; coin B has mass 10 g and is placed 7.5 cm from the centre. The coefficient of friction between each coin and the turntable is 0.4.

(i) Describe what happens to the coins when the turntable turns at

(a) 6 rad s^{-1} (b) 8 rad s^{-1} (c) 10 rad s^{-1}.

(ii) What would happen if the coins were interchanged?

③ A car is travelling at a steady speed of $15\,\mathrm{m\,s^{-1}}$ round a roundabout of radius 20 m on a flat horizontal road.

(i) Criticise this false argument:

The car is travelling at a steady speed and so its speed is neither increasing nor decreasing and therefore the car has no acceleration.

(ii) Calculate the magnitude of the acceleration of the car.

(iii) The car has mass 800 kg. Calculate the sideways force on each wheel assuming it to be the same for all four wheels.

(iv) Is the assumption in part (iii) realistic?

④ A fairground ride has seats at 3 m and at 4.5 m from the centre of rotation. Each rider travels in a horizontal circle. Say whether each of the following statements is TRUE or FALSE, giving your reasons.

(i) Riders in the two positions have the same angular speed at any time.

(ii) Riders in the two positions have the same speed at any time.

(iii) Riders in the two positions have the same magnitude of acceleration at any time.

⑤ A particle C of mass 3.2 kg is moving in a horizontal circle of radius r m.

The centre of the circle is O and C moves with a constant angular speed of exactly 8 radians per second.

The particle is attached to points A and B by two light inextensible strings which are taut.

As shown in the diagram the points O, A and B lie on the axis of rotation.

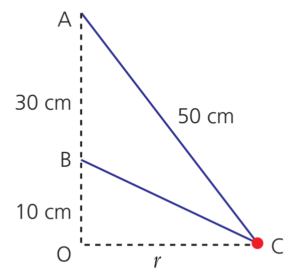


Figure 7.25

AC = 50 cm, AB = 30 cm and BO = 10 cm.

Taking the acceleration due to gravity to be $9.8\,\mathrm{m\,s^{-2}}$, find the tensions in the two strings.

⑥ Two spin driers, both of which rotate about a vertical axis, have different specifications as given in the table below.

Model	Rate of rotation	Drum diameter
A	600 rpm	60 cm
B	800 rpm	40 cm

State with reasons, which model you would expect to be the more effective.

⑦ A satellite of mass M_s, is in a circular orbit around the Earth, with a radius of r metres. The force of attraction between the Earth and the satellite is given by

$$F = \frac{GM_eM_s}{r^2}$$

where $G = 6.67 \times 10^{-11}$ in S.I. units. The mass of the earth M_e is 5.97×10^{24} kg.

(i) Find in terms of r, expressions for

(a) the speed of the satellite, $v\,\mathrm{m\,s^{-1}}$

(b) the time T s, it takes to complete one revolution.

(ii) Hence show that, for all satellites, T^2 is proportional to r^3.

Note

The law found in part (ii) was discovered experimentally by Johannes Kepler (1571–1630) to hold true for the planets as they orbit the Sun, and is commonly known as Kepler's third law.

A geostationary satellite orbits the Earth so that it is always above the same place on the equator.

(iii) How far is it from the centre of the Earth?

⑧ A rotary lawn mower uses a piece of light nylon string with a small metal sphere on the end to cut the grass. The string is 20 cm in length and the mass of the sphere is 30 g.

(i) Find the tension in the string when the sphere is rotating at 2000 rpm assuming that the string is horizontal.

(ii) Explain why it is reasonable to assume that the string is horizontal.

(iii) Find the speed of the sphere when the tension in the string is 80 N.

⑨ In this question, you should assume that the orbit of the Earth around the Sun is circular, with radius 1.44×10^{11} m, and that the Sun is fixed.

(i) Find the magnitude of the acceleration of the Earth as it orbits the Sun.

The force of attraction between the Earth and the Sun is given by

$$F = \frac{GM_eM_s}{r^2}$$

where M_e is the mass of the Earth, M_s is the mass of the Sun, r is the radius of the Earth's orbit and G the universal constant of gravitation (6.67×10^{-11} S.I. units).

(ii) Calculate the mass of the Sun.

(iii) Comment on the significance of the fact that you cannot calculate the mass of the Earth from the radius of its orbit.

⑩ Experiments carried out by the police accident investigation department suggest that a typical value for a coefficient of friction between the tyres of a car and a road surface is 0.8.

(i) Using this information, find the maximum safe speed on a level circular motorway slip road of radius 50 m.

(ii) How much faster could cars travel if the road were banked at an angle of 5° to the horizontal?

⑪ A light inextensible string of length 0.8 m is threaded through a smooth ring and carries a particle at each end. Particle A of mass m kg is at rest at a distance of 0.3 m below the ring. The other particle, B, of mass M kg is rotating in a horizontal circle whose centre is A.

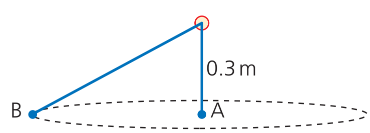


Figure 7.26

(i) Express M in terms of m.

(ii) Find the angular velocity of B.

⑫ A particle of mass 0.2 kg is moving on the smooth inside surface of a fixed hollow sphere of radius 0.75 m. The particle moves in a horizontal circle whose centre is 0.45 m below the centre of the sphere (see Figure 7.24).

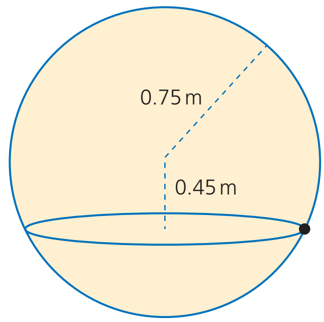


Figure 7.27

(i) Show that the force exerted by the sphere on the particle has magnitude $\frac{1}{3}g$.

(ii) Find the speed of the particle.

(iii) Find the time taken for the particle to complete one revolution.

⑬ A particle P of mass 0.25 kg is attached to one end of each of two inextensible strings which are both taut. The other end of the longer string is attached to a fixed point A, and the other end of the shorter string is attached to a fixed point B, which is vertically below A.

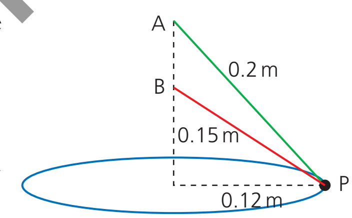


Figure 7.28

String AP is 0.2 m long and string BP is 0.15 m long. P moves in a horizontal circle of radius 0.12 m with constant angular speed 10 rad s^{-1}. Both strings are taut: T_1 is the tension in AP and T_2 is the tension in BP.

(i) Resolve vertically to show that $8T_1 + 6T_2 = 2.5g$.

(ii) Find another equation connecting T_1 and T_2 and hence calculate T_1 and T_2.

⑭ A particle P of mass m kg is moving in a horizontal circle of radius r m. The centre of the circle is O and C moves with a constant angular speed of exactly ω radians per second.

The particle is attached to points Q and R by two light inextensible strings. The points O, Q and R lie on a vertical line. The strings RP and QP make angles α and β with the horizontal. The tension in string RP is T_1 and that in QP is T_2.

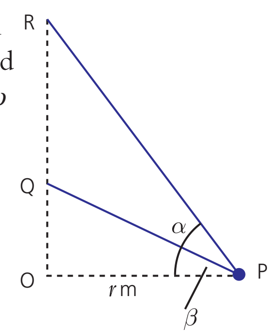


Figure 7.29

(i) Show that when the strings are taut
$T_1 = \dfrac{m\left(g\cos\beta - \omega^2 r\sin\beta\right)}{\sin(\alpha - \beta)}$ and find an equivalent expression for T_2.

(ii) What are the implications if the value of T_2 is zero?

⑮ A particle C of mass 2.0 kg is moving in a horizontal circle of radius r m.

The centre of the circle is O and C moves with a constant angular speed of exactly 4 radians per second.

The particle is attached to points A and B by two light inextensible strings which are taut.

As shown in Figure 7.30, the points O, A and B lie on the axis of rotation. The point B is below O.

AO = 0.90 m, OB = 0.40 m and angle ACB is a right angle.

Taking g to be 10 m s^{-2}, find the tensions in the two strings.

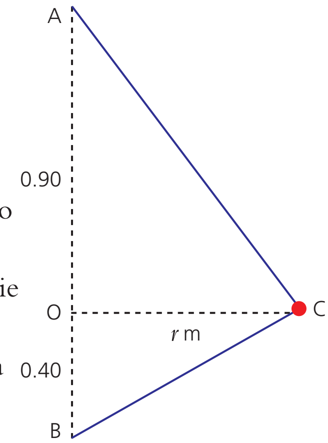


Figure 7.30

KEY POINTS

1 Position, velocity and acceleration of a particle moving on a circle of radius r.

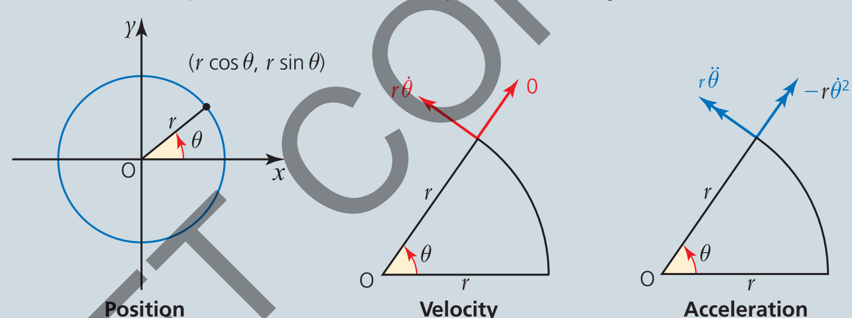


Figure 7.31

- position — $(r\cos\theta, r\sin\theta)$
- velocity — transverse component: $v = r\dot{\theta} = r\omega$
  radial component: 0
  where $\dot{\theta}$ or ω is the angular velocity of the particle.
- acceleration — transverse component: $r\ddot{\theta} = r\dot{\omega}$
  radial component: $-r\dot{\theta}^2 = -r\omega^2 = -\frac{v^2}{r}$
  where $\ddot{\theta}$ or $\dot{\omega}$ is the angular acceleration of the particle.

2 By Newton's second law, the forces acting on a particle of mass m in circular motion are equal to

- transverse component: $mr\dot{\omega} = mr\ddot{\theta}$
- radial component: $-\frac{mv^2}{r} = -mr\omega^2$
- or radial component: $+\frac{mv^2}{r} = +mr\omega^2$ towards the centre

LEARNING OUTCOMES

When you have finished this chapter, you should be able to:

- understand the language and notation associated with circular motion
- identify the forces acting on a body in circular motion
- calculate acceleration towards the centre of circular motion
- model situations involving circular motion with uniform speed in a horizontal plane.